On mappings in the Orlicz-Sobolev classes

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IN MEMORY OF ALBERTO CALDERON (1920–1998)

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Abstract

First of all, we prove that open mappings in Orlicz-Sobolev classes $W^{1,\varphi}_{\rm loc}$ under the Calderon type condition on φ have the total differential a.e. that is a generalization of the well-known theorems of Gehring-Lehto-Menchoff in the plane and of Väisälä in $\mathbb{R}^n$, $n \geqslant 3$. Under the same condition on φ, we show that continuous mappings f in $W^{1,\varphi}_{\rm loc}$, in particular, $f \in W^{1,p}_{\rm loc}$ for $p > n-1$ have the (N)-property by Lusin on a.e. hyperplane. Our examples demonstrate that the Calderon type condition is not only sufficient but also necessary for this and, in particular, there exist homeomorphisms in $W^{1,n-1}_{\rm loc}$ which have not the (N)-property with respect to the $(n-1)$-dimensional Hausdorff measure on a.e. hyperplane. It is proved on this base that under this condition on φ the homeomorphisms f with finite distortion in $W^{1,\varphi}_{\rm loc}$ and, in particular, $f \in W^{1,p}_{\rm loc}$ for $p > n-1$ are the so-called lower Q-homeomorphisms where $Q(x)$ is equal to its outer dilatation $K_f(x)$ as well as the so-called ring Q_*-homeomorphisms with $Q_*(x) = [K_f(x)]^{n-1}$. This makes possible to apply our theory of the local and boundary behavior of the lower and ring Q-homeomorphisms to homeomorphisms with finite distortion in the Orlicz-Sobolev classes.

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Contents

1 Introduction

Moduli provide us the main geometric tool in the mapping theory. The recent development of the moduli method in the connection with modern classes of mappings can be found in the monograph [115], see also recent books in the moduli and capacity theory [7], [27] and [176] as well as the following papers and monographs [5], [79], [101], [161], [163], [169], [185] and further references therein. In the present paper we show that the theories of the so-called lower and ring Q-homeomorphisms developed in [115] can be applied to a wide range of mappings with finite distortion in the Orlicz-Sobolev classes. Note that the plane case has been recently studied in [84] and [106]. Recall, it was established therein that each homeomorphism of finite distortion in the plane is a lower and ring Q-homeomorphism with $Q(x) = K_f(x)$.

In what follows, D is a domain in a finite-dimensional Euclidean space. Following Orlicz, see [126], given a convex increasing function $\varphi : [0,\infty) \to [0,\infty)$, $\varphi(0) = 0$, denote by L_φ the space of all functions $f : D \to \mathbb{R}$ such that

$$\int\limits_D \varphi\left(\frac{|f(x)|}{\lambda}\right) dm(x) < \infty \tag{1.1}$$

for some $\lambda > 0$ where $dm(x)$ corresponds to the Lebesgue measure in D, see also the monographs [86] and [181]. L_φ is called the **Orlicz space**. If $\varphi(t) = t^p$, then we write L_p. In other words, L_φ is the cone over the class of all functions $g : D \to \mathbb{R}$ such that

$$\int\limits_D \varphi\left(|g(x)|\right) dm(x) < \infty \tag{1.2}$$

which is also called the **Orlicz class**, see [14].

The **Orlicz-Sobolev class** $W^{1,\varphi}_{\rm loc}(D)$ is the class of all locally integrable functions f given in D with the first distributional derivatives whose gradient ∇f belongs locally in D to the Orlicz class. Note that by definition $W^{1,\varphi}_{\rm loc}\subseteq W^{1,1}_{\rm loc}$. As usual, we write $f\in W^{1,p}_{\rm loc}$ if $\varphi(t)=t^p$, $p\geqslant 1$. It is known that a continuous function f belongs to $W^{1,p}_{\rm loc}$ if and only if $f\in ACL^p$, i.e., if f is locally absolutely continuous on a.e. straight line which is parallel to a coordinate axis and if the first partial derivatives of f are locally integrable with the power p, see, e.g., 1.1.3 in [119]. The concept of the distributional derivative was introduced by Sobolev [162] in $\mathbb{R}^n$, $n\geqslant 2$, and now it is developed under wider settings, see, e.g., [3], [47], [50], [52], [54], [110], [115], [146], [171] and [172].

Later on, we also write $f\in W^{1,\varphi}_{\rm loc}$ for a locally integrable vector-function $f=(f_1,\ldots,f_m)$ of n real variables $x_1,\ldots,x_n$ if $f_i\in W^{1,1}_{\rm loc}$ and

$$\int\limits_{D}\varphi\left(|\nabla f(x)|\right)\,dm(x)<\infty \tag{1.3}$$

where $|\nabla f(x)|=\sqrt{\sum\limits_{i,j}\left(\frac{\partial f_i}{\partial x_j}\right)^2}$. In the main part of the paper we use the notation $W^{1,\varphi}_{\rm loc}$ for more general functions φ than in the classical Orlicz classes giving up the condition on convexity of φ. In fact we need the convexity of φ only in Section 13. Note that the Orlicz-Sobolev classes are intensively studied in various aspects, see, e.g., [2], [5], [13], [16], [18], [26], [41], [60], [64], [74], [85], [102], [103], [170] and [178].

Recall that a homeomorphism f between domains D and D' in $\mathbb{R}^n$, $n\geqslant 2$, is called of **finite distortion** if $f\in W^{1,1}_{\rm loc}$ and

$$\|f'(x)\|^n\leqslant K(x)\cdot J_f(x) \tag{1.4}$$

with a.e. finite function K where $\|f'(x)\|$ denotes the matrix norm of the Jacobian matrix f' of f at $x\in D$, $||f'(x)||=\sup\limits_{h\in\mathbb{R}^n,|h|=1}|f'(x)\cdot h|$,

and $J_f(x) = \det f'(x)$ is its Jacobian. Later on, we use the notation $K_f(x)$ for the minimal function $K(x) \geqslant 1$ in (1.4), i.e., we set $K_f(x) = \|f'(x)\|^n / \, J_f(x)$ if $J_f(x) \neq 0$, $K_f(x) = 1$ if $f'(x) = 0$ and $K_f(x) = \infty$ at the rest points.

First this notion was introduced on the plane for $f \in W^{1,2}_{\rm loc}$ in the work [66]. Later on, this condition was changed by $f \in W^{1,1}_{\rm loc}$ but with the additional condition $J_f \in L^1_{\rm loc}$ in the monograph [65]. The theory of the mappings with finite distortion had many successors, see, e.g., [6], [20], [21], [30], [35], [53]–[56], [59], [63], [64], [68], [71], [72], [73], [75]–[78], [108], [109], [123]–[125], [129] and [137]–[140]. They had as predecessors the mappings with bounded distortion, see [144] and [177], in other words, the quasiregular mappings, see, e.g., [51], [111] and [148]. They are also closely related to the earlier mappings with the bounded Dirichlet integral, see, e.g., [105] and [166]–[168], and the mappings quasiconformal in the mean which had a rich history, see, e.g., [1], [11], [12], [39], [40], [44], [46], [87]–[99], [130], [131], [132], [149]–[151], [155], [164], [165], [172], [183] and [184].

Note that the above additional condition $J_f \in L^1_{\rm loc}$ in the definition of the mappings with finite distortion can be omitted for homeomorphisms. Indeed, for each homeomorphism f between domains D and D' in $\mathbb{R}^n$ with the first partial derivatives a.e. in D, there is a set E of the Lebesgue measure zero such that f satisfies (N)-property by Lusin on $D \setminus E$ and

$$\int\limits_A J_f(x)\, dm(x) = |f(A)| \tag{1.5}$$

for every Borel set $A \subset D \setminus E$, see, e.g., 3.1.4, 3.1.8 and 3.2.5 in [31]. On this base, it is easy by the Hölder inequality to verify, in particular, that if $f \in W^{1,1}_{\rm loc}$ is a homeomorphism and $K_f \in L^q_{\rm loc}$ for $q > n-1$, then also $f \in W^{1,p}_{\rm loc}$ for $p > n-1$, that we often use further to obtain corollaries.

In this paper H^k, $k \geqslant 0$, denotes the **k-dimensional Hausdorff measure** in $\mathbb{R}^n$, $n \geqslant 1$. More precisely, if A is a set in $\mathbb{R}^n$, then

$$H^k(A) \;=\; \sup_{\varepsilon>0} \; H^k_\varepsilon(A)\,, \tag{1.6}$$

$$H^k_\varepsilon(A) \;=\; \inf \sum_{i=1}^{\infty} (\operatorname{diam} A_i)^k \;, \tag{1.7}$$

where the infimum in (1.7) is taken over all coverings of A by sets A_i with $\operatorname{diam} A_i < \varepsilon$, see, e.g., [118] in this connection. Note that H^k is an **outer measure in the sense of Caratheodory**, i.e.,

(1) $H^k(X) \;\leqslant\; H^k(Y)$ whenever $X \subseteq Y$,

(2) $H^k(\bigcup\limits_i X_i) \;\leqslant\; \sum\limits_i H^k(X_i)$ for each sequence of sets X_i,

(3) $H^k(X \cup Y) \;=\; H^k(X) + H^k(Y)$ whenever $\operatorname{dist}(X,Y) > 0$.

A set $E \subset \mathbb{R}^n$ is called **measurable** with respect to H^k if $H^k(X) = H^k(X \cap E) + H^k(X \setminus E)$ for every set $X \subset \mathbb{R}^n$. It is well known that every Borel set is measurable with respect to any outer measure in the sense of Caratheodory, see, e.g., Theorem II (7.4) in [156]. Moreover, H^k is Borel regular, i.e., for every set $X \subset \mathbb{R}^n$, there is a Borel set $B \subset \mathbb{R}^n$ such that $X \subset B$ and $H^k(X) = H^k(B)$, see, e.g., Theorem II (8.1) in [156] and Section 2.10.1 in [31]. The latter implies that, for every measurable set $E \subset \mathbb{R}^n$, there exist Borel sets B_* and $B^* \subset \mathbb{R}^n$ such that $B_* \subset E \subset B^*$ and $H^k(B^* \setminus B_*) = 0$, see, e.g., Section 2.2.3 in [31]. In particular, $H^k(B^*) = H^k(E) = H^k(B_*)$.

If $H^{k_1}(A) < \infty$, then $H^{k_2}(A) = 0$ for every $k_2 > k_1$, see, e.g., VII.1.B in [61]. The quantity

$$\dim_H A = \sup_{H^k(A)>0} k$$

is called the **Hausdorff dimension** of a set A.

It is known that the outer Lebesgue measure $m(A) = \Omega_n {\cdot} 2^{-n} H^n(A)$ for sets A in $\mathbb{R}^n$ where Ω_n denotes the volume of the unit ball in $\mathbb{R}^n$, see [158].

It was shown in [38] that a set A with $\dim_H A = p$ can be transformed into a set $B = f(A)$ with $\dim_H B = q$ for each pair of numbers p and $q \in (0, n)$ under a quasiconformal mapping f of $\mathbb{R}^n$ onto itself, cf. also [8] and [13].

2 Preliminaries

First of all, the following fine property of functions f in the Sobolev classes $W^{1,p}_{\rm loc}$ was proved in the monograph [41], Theorem 5.5, and can be extended to the Orlicz-Sobolev classes. The statement follows directly from the Fubini theorem and the known characterization of functions in Sobolev's class $W^{1,1}_{\rm loc}$ in terms of ACL (absolute continuity on lines), see, e.g., Section 1.1.3 in [119], and comments in Introduction.

Proposition 2.1. *Let U be an open set in $\mathbb{R}^n$ and let $f : U \to \mathbb{R}^m$, $m = 1, 2, \ldots$, be a mapping in the Orlicz-Sobolev class $W^{1,\varphi}_{\rm loc}(U)$ with an increasing function $\varphi : [0, \infty) \to [0, \infty)$. Then, for every k-dimensional direction Γ for a.e. k-dimensional plane $\mathcal{P} \in \Gamma$, $k = 1, 2, \ldots, n-1$, the restriction of the function f on the set $\mathcal{P} \cap U$ is a function in the class $W^{1,\varphi}_{\rm loc}(\mathcal{P} \cap U)$.*

Here the class $W^{1,\varphi}_{\rm loc}$ is well defined on a.e. k-dimensional plane because partial derivatives are Borel functions and, moreover, Sobolev's classes are invariant with respect to quasi-isometric transformations of systems of coordinates, in particular, with respect to rotations, see, e.g., 1.1.7 in [119]. Recall also that a k-dimensional direction Γ in $\mathbb{R}^n$ is the class of equivalence of all k-dimensional planes in $\mathbb{R}^n$ that can be obtained each from other by a parallel shift. Note that each $(n-k)$-dimensional plane $\mathcal{T}$ which is quite orthogonal to a k-dimensional plane $\mathcal{P}$ in Γ intersects $\mathcal{P}$ at a single point $X(\mathcal{P})$. If E is a subset of Γ, then $X(E)$ denotes the collection of all point $X(\mathcal{P})$, $\mathcal{P} \in E$. It is clear that $(n-k)$-dimensional measure of the set $X(E)$ does not depend of the choice of the plane $\mathcal{T}$ and it is denoted by $\mu_{n-k}(E)$. They say that a property holds for almost every plane in

Γ if $\mu_{n-k}(E) = 0$ for a set E of all planes $\mathcal{P}$ for which the property fails.

Recall also the little–known Fadell theorem in [29] that makes possible us to extend the well-known theorems of Gehring-Lehto-Menchoff in the plane and Väisälä in $\mathbb{R}^n$, $n \geqslant 3$, see, e.g., [36], [120] and [174], on differentiability a.e. of open mappings in Sobolev's classes to the open mappings in Orlicz-Sobolev classes in $\mathbb{R}^n$, $n \geqslant 3$.

Proposition 2.2. *Let $f : \Omega \to \mathbb{R}^n$ be a continuous open mapping on an open set Ω in $\mathbb{R}^n$, $n \geqslant 3$. If f has a total differential a.e. on Ω with respect to $n-1$ variables, then f has a total differential a.e. on Ω.*

Now, let us give the following Calderon result in [16], p. 208, cf. Lemma 3.2 in [70].

Proposition 2.3. *Let $\varphi : [0, \infty) \to [0, \infty)$ be an increasing function such that $\varphi(0) = 0$ and*

$$A \ : \ = \int\limits_0^{\infty} \left[\frac{t}{\varphi(t)}\right]^{\frac{1}{k-1}} \ dt \ < \ \infty \tag{2.1}$$

for a natural number $k \geqslant 2$ and let $f : G \to \mathbb{R}$ be a function given in a domain $G \subset \mathbb{R}^k$ of the class $W^{1,\varphi}(G)$. Then

$$\operatorname{diam} f(C) \leqslant \alpha_k A^{\frac{k-1}{k}} \left[\ \int\limits_C \varphi\left(|\nabla f|\right) \, dm(x)\right]^{\frac{1}{k}} \tag{2.2}$$

for every cube $C \subset G$ whose adges are oriented along coordinate axes where α_k is a constant depending only on k.

Perhaps, the Calderon work [16] had time to be forgotten because it was published long ago in a badly accessible journal.

Remark 2.1. It is clear that the behavior of the function φ about 0 is not essential and (2.2) holds with the replacement of the constant

A by the constant

$$A_* \ := \ \left[\frac{1}{\varphi(1)}\right]^{\frac{1}{k-1}} \ + \ \int\limits_1^{\infty} \left[\frac{t}{\varphi(t)}\right]^{\frac{1}{k-1}} \, dt \ < \ \infty \tag{2.3}$$

and $\varphi(t)$ by $\varphi_*(t) \equiv \varphi(1)$ for $t \in (0,1)$, $\varphi_*(0) = 0$ and $\varphi_*(t) = \varphi(t)$ for $t \geqslant 1$. Indeed, applying Proposition 2.3 to the one parameter family of the functions $\varphi_\lambda(t) = \varphi(t) + \lambda \cdot [\varphi_*(t) - \varphi(t)]$, $\lambda \in [0,1)$, we obtain (2.2) with the changes $A \mapsto A_*$ and $\varphi \mapsto \varphi_*$ as $\lambda \to 1$.

Finally, one more statement of Calderon in [16], p. 209, 211-212, will be also useful later on.

Proposition 2.4. *Let $\varphi : [0,\infty) \to [0,\infty)$ be a convex increasing function such that $\varphi(0) = 0$ and*

$$\int\limits_1^{\infty} \left[\frac{t}{\varphi(t)}\right]^{\frac{1}{k-1}} dt = \infty \tag{2.4}$$

for a natural number $k \geqslant 2$. Then there is a continuously differentiable decreasing function $F : (0,\infty) \to [0,\infty)$ with a compact support such that $F(t) \to \infty$ as $t \to 0$, $F'(t)$ is non-decreasing, $F'(t) \to -\infty$ as $t \to 0$ and $F_*(x) = F(|x|)$, $x \in \mathbb{R}^k$, belongs to the class $W^{1,\varphi}(\mathbb{R}^k)$, i.e., $f \in W^{1,1}(\mathbb{R}^k)$ and

$$\int\limits_{\mathbb{R}^k} \varphi\left(|\nabla F_*|\right) \, dm(x) \leqslant 1. \tag{2.5}$$

Remark 2.2. The function F from Proposition 2.4 can be described in a more constructive way. More precisely, set

$$\Phi(t) \ = \ \int\limits_1^{t} \left[\frac{\tau}{\varphi(\tau)}\right]^{\frac{1}{k-1}} \ d\tau \tag{2.6}$$

and

$$\Psi(t) \ = \ \frac{\Phi'(t)}{\Phi(t)} \ = \ \left[\frac{t}{\varphi(t)}\right]^{\frac{1}{k-1}} \frac{1}{\Phi(t)}. \tag{2.7}$$

The function Ψ is continuous and decreasing and tends to 0 as $t \to \infty$ and to ∞ as $t \to 1$. thus, its inverse function $h(s)$ is well defined for all $s > 0$. It was proved by Calderon in [16] that

$$\int\limits_{0}^{1} h(s)\, ds \;=\; \infty\,, \qquad \int\limits_{0}^{1} \varphi(h(s))\, s^{k-1}\, ds < \infty\,. \tag{2.8}$$

Then we may set $F(t) \equiv 0$ for $t \geqslant 1$ and

$$F(t) = \int\limits_{t}^{1} [h(s) - h(1)]\, ds \qquad \forall\, t \in [0,1]\,. \tag{2.9}$$

On the base of Proposition 2.3, it was proved by Calderon that each continuous function $f : G \to \mathbb{R}$ in $W^{1,\varphi}(G)$ under the condition

$$\int\limits_{1}^{\infty} \left[\frac{t}{\varphi(t)}\right]^{\frac{1}{k-1}} dt < \infty \tag{2.10}$$

has a total differential a.e. Moreover, on the base of Proposition 2.4, under the condition (2.4) Calderon has constructed a continuous function $f : \mathbb{R}^k \to \mathbb{R}$ which has not a total differential a.e. We use Propositions 2.3 and 2.4 for other purposes.

3 The differentiability of open mappings

Let us start from the following statement which is due to Calderon [16] but we prefer in comparison with [16] to prove it on the base of the Stepanoff theorem.

Lemma 3.1. *Let Ω be an open set in $\mathbb{R}^k$, $k \geqslant 2$, and let $f : \Omega \to \mathbb{R}^m$, $m \geq 1$, be a continuous mapping in the class $W^{1,\varphi}_{\rm loc}(\Omega)$ with some increasing $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(0) = 0$ and*

$$A := \int\limits_{1}^{\infty} \left[\frac{t}{\varphi(t)}\right]^{\frac{1}{k-1}} dt < \infty. \tag{3.1}$$

Then f has a total differential a.e. in Ω.

Proof. Given $x \in \Omega$, we set

$$L(x,f) = \limsup_{y\to x} \frac{|f(y)-f(x)|}{|y-x|}\,.$$

By the Stepanoff theorem, see, e.g., Theorem 3.1.9 in [31], the proof is reduced to the proof of the fact $L(x,f) < \infty$ a.e. in Ω.

Denote by $C(x,r)$ the oriented cube centered at x such that the ball $B(x,r)$ is inscribed in $C(x,r)$ where $r = |x-y|$. Then

$$L(x,f) \;=\; \limsup_{y\to x} \frac{|f(x)-f(y)|}{|x-y|} \;\leqslant$$

$$\leqslant \limsup_{r\to 0} \frac{d(fB(x,r))}{r} \;\leqslant\; \limsup_{r\to 0} \frac{d(fC(x,r))}{r}$$

and by Proposition 2.3 and Remark 2.1 we get

$$L(x,f) \leqslant \gamma_{k,m}\, A_*^{\frac{k-1}{k}} \limsup_{r\to 0} \left[\frac{1}{r^k}\int\limits_{C(x,r)} \varphi_*\left(|\nabla f|\right)\, dm(x)\right]^{\frac{1}{k}} < \infty$$

for a.e. $x \in \Omega$ by the Lebesgue theorem on differentiability of indefinite integral, see, e.g., Theorem IV.5.4 in [156]. The proof is complete.

Combining Lemma 3.1 and Proposition 2.1, we obtain the following statement.

Corollary 3.1. *Let Ω be an open set in $\mathbb{R}^n$, $n \geqslant 3$, and let $f : \Omega \to \mathbb{R}^m$, $m \geq 1$, be a continuous mapping in the class $W^{1,\varphi}_{\rm loc}(\Omega)$ with an increasing function $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(0) = 0$ and*

$$\int\limits_1^{\infty} \left[\frac{t}{\varphi(t)}\right]^{\frac{1}{n-2}} dt < \infty. \tag{3.2}$$

Then $f : \Omega \to \mathbb{R}^m$ has a total differential a.e. on a.e. hyperplane which is parallel to a coordinate hyperplane.

Combing Corollary 3.1 and the Fadell result in [29], see Proposition 2.2 above, we obtain the main result of this section.

Theorem 3.1. *Let Ω be an open set in $\mathbb{R}^n$, $n \geqslant 3$, and let $f : \Omega \to \mathbb{R}^n$ be a continuous open mapping in the class $W^{1,\varphi}_{\mathrm{loc}}(\Omega)$ with an increasing $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(0) = 0$ and (3.2) holds. Then f has a total differential a.e. in Ω.*

Corollary 3.2. *If $f : \Omega \to \mathbb{R}^n$ is a homeomorphism in $W^{1,1}_{\mathrm{loc}}$ with $K_f \in L^p_{\mathrm{loc}}$ for $p > n-1$, then f is differentiable a.e.*

Remark 3.1. In particular, the conclusion is true if $f \in W^{1,p}_{\mathrm{loc}}$ for some $p > n-1$. The latter statement is the Väisälä result, see Lemma 3 in [174]. Theorem 3.1 is also an extension of the well-known Gegring-Lehto-Menchoff result in the plane to high dimensions, see, e.g., [36], [104] and [120].

Calderon has shown in [16] the preciseness of the condition of (3.1) for differentiability a.e. of continuous mappings f. Theorem 3.1 shows that we may use the weaker condition (3.2) to obtain differentiability a.e. of open mappings f.

The condition (3.2) is not only sufficient but also necessary for open continuous mappings f $W^{1,\varphi}_{\mathrm{loc}}$ from $\mathbb{R}^n$ into $\mathbb{R}^n$, $n \geqslant 3$, to have a total differential a.e. Furthemore, if a function $\varphi : [0,\infty) \to [0,\infty)$ is increasing, convex and such that

$$\int\limits_1^{\infty} \left[\frac{t}{\varphi(t)}\right]^{\frac{1}{n-2}} dt = \infty \ , \tag{3.3}$$

then there is a homeomorphism $g : \mathbb{R}^n \to \mathbb{R}^n$, $n \geqslant 3$, in the class $W^{1,\varphi}_{\mathrm{loc}}$ which has not a total differential a.e. Indeed, if $f : \mathbb{R}^{n-1} \to \mathbb{R}$ is a function in the Calderon construction for $k = n-1$ and $\varphi_*(t) = \varphi(t+k) - \varphi(k)$, then

$$\int\limits_1^{\infty} \left[\frac{t}{\varphi_*(t)}\right]^{\frac{1}{n-2}} dt = \infty \tag{3.4}$$

and $g(x,y) = (x, y+f(x))$, $x \in \mathbb{R}^{n-1}, y \in \mathbb{R}$, is the desired example because of $|\nabla g| \leqslant k + |\nabla f|$ and monotonicity of the function φ. Thus, the condition (3.2) already cannot be weakened even for homeomorphisms.

4 The Lusin and Sard properties on surfaces

Theorem 4.1. *Let Ω be an open set in $\mathbb{R}^k$, $k \geqslant 2$, and let $f : \Omega \to \mathbb{R}^m$, $m \geq 1$, be a continuous mapping in the class $W^{1,\varphi}(\Omega)$ with an increasing $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(0) = 0$ and*

$$A := \int\limits_1^\infty \left[\frac{t}{\varphi(t)}\right]^{\frac{1}{k-1}} dt < \infty\,. \tag{4.1}$$

Then

$$H^k(f(E)) \leqslant \gamma_{k,m}\, A_*^{k-1} \int\limits_E \varphi_*\left(|\nabla f|\right)\, dm(x) \tag{4.2}$$

for every measurable set $E \subset \Omega$ and $\gamma_{k,m} = (m\alpha_k)^k$ where α_k is a constant from (2.2) depending only on k, $A_* = A + 1/[\varphi(1)]^{1/(k-1)}$, $\varphi_*(0) = 0$, $\varphi_*(t) \equiv \varphi(1)$ for $t \in (0,1)$ and $\varphi_*(t) = \varphi(t)$ for $t \geqslant 1$.

Thus, we obtain the following conclusions on the Lusin property of mappings in the Orlicz-Sobolev classes.

Corollary 4.1. *Under the hypotheses of Theorem 4.1 the mapping f has the (N)-property of Lusin (furthermore, f is absolutely continuous) with respect to the k-dimensional Hausdorff measure.*

Remark 4.1. Note that $H^k(\mathbb{R}^m) = 0$ for $m < k$ and hence (4.2) is trivial in this case without the condition (4.1). However, the examples in Section 13 show that the condition (4.1) is not only sufficient but also necessary for the (N)–property if $m \geq k$, see Lemma 13.1 and Remark 13.2.

We obtain also the following consequence of Theorem 4.1 of the Sard type for mappings in the Orlicz-Sobolev classes, see in addition Theorem VII.3 in [61].

Corollary 4.2. *Under the hypotheses of Theorem 4.1, we have that $H^k(f(E)) = 0$ whenever $|\nabla f| = 0$ on a measurable set $E \subset \Omega$ and hence* $\dim_H f(E) \leqslant k$ and also $\dim f(E) \leqslant k-1$.

First such a statement was established by Sard in [157] for the set of **critical points** of f where $J_f(x) = 0$ and then similar problems studied by many authors for **critical points of ranks** r where rank $f'(x) \leqslant r$ and, in particular, for **supercritical points** where the Jacobian matrix $f'(x)$ is null at all, see, e.g., [10], [22], [24], [25], [28], [43], [48], [69], [122], [134], [159], [160] and [180]. Usually they requested the corresponding conditions of smoothness for f without which such statements, generally speaking, are not true.

In this connection, we would like to stress here that our result on supercritical points, Corollary 4.2, holds without any assumptions on smoothness of f. For instance, this result holds for all continuous mappings f in the class $W^{1,p}_{\rm loc}$ with $p > k$, see a fine survey on Sard's theorems, in particular, for Sobolev mappings in the paper [15].

The proof of Theorem 4.1 is based on the following lemma.

Lemma 4.1. *Let Ω be a domain in $\mathbb{R}^k$, $k \geqslant 2$, and let $f : \Omega \to \mathbb{R}^m$, $m \geq 1$, be a continuous mapping in the class $W^{1,\varphi}(G)$ with an increasing $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(0) = 0$ and*

$$A := \int\limits_1^{\infty} \left[\frac{t}{\varphi(t)}\right]^{\frac{1}{k-1}} dt < \infty. \tag{4.3}$$

Then

$$\operatorname{diam} f(C) \leqslant m\alpha_k\, A_*^{\frac{k-1}{k}} \left[\int\limits_C \varphi_*\left(|\nabla f|\right)\, dm(x)\right]^{\frac{1}{k}} \tag{4.4}$$

for every cube $C \subset \Omega$ whose adges are oriented along coordinate axes where α_k is a constant from (2.2) depending only on k and A_* and φ_* have been defined in Theorem 4.1.

Proof of Lemma 4.1. Let us prove (4.4) by induction in $m = 1, 2, \ldots$. Indeed, (4.4) holds by Proposition 2.3 and Remark 2.1 for $m = 1$ and α_k from (2.2). Let us assume that (4.4) is valid for some $m = l$ and prove it for $m = l + 1$. Consider an arbitrary vector $\vec{V} = (v_1, v_2, \ldots, v_l, v_{l+1})$ in $\mathbb{R}^{l+1}$ and the vectors $\vec{V_1} = (v_1, v_2, \ldots, v_l, 0)$ and $\vec{V_2} = (0, \ldots, 0, v_{k+1})$. Then $|\vec{V}| = |\vec{V_1} + \vec{V_2}| \leqslant |\vec{V_1}| + |\vec{V_2}|$. Thus, denoting by $\mathrm{Pr}_1 \vec{V} = \vec{V_1}$ and $\mathrm{Pr}_2 \vec{V} = \vec{V_2}$ the projections of vectors from $\mathbb{R}^{l+1}$ onto the coordinate hyperplane $y_{l+1} = 0$ and on the $(l+1)$th axis in $\mathbb{R}^{l+1}$, correspondingly, we obtain that $\operatorname{diam} f(C) \leqslant \operatorname{diam} \mathrm{Pr}_1 f(C) + \operatorname{diam} \mathrm{Pr}_2 f(C)$ and, applying (4.4) under $m = l$ and $m = 1$, we come by monotonicity of φ to the inequality (4.4) under $m = l + 1$. The proof is complete.

Proof of Theorem 4.1. In view of countable additivity of integral and measure we may assume with no loss of generality that E is bounded and $\overline{E} \subset G$, i.e., $\overline{E}$ is a compactum in G. For each $\varepsilon > 0$ there is an open set $\Omega \subset G$ such that $E \subset \Omega$ and $|\Omega \setminus E| < \varepsilon$, see, e.g., Theorem III (6.6) in [156]. By the above remark we may assume that $\overline{\Omega}$ is a compactum and, thus, the mapping f is uniformly continuous in Ω. Hence Ω can be covered by a countable collection of closed oriented cubes C_i whose interiorities are mutually disjoint and such that $\operatorname{diam} f(C_i) < \delta$ for any prescribed $\delta > 0$ and $\left|\bigcup\limits_{i=1}^{\infty} \partial C_i\right| = 0$.

Thus, by Lemma 4.1 we have that

$$H^k_\delta(f(E)) \leqslant H^k_\delta(f(\Omega)) \leqslant \sum_{i=1}^{\infty} [\operatorname{diam} f(C_i)]^k \leqslant$$

$$\leqslant \gamma_{k,m} A_*^{k-1} \int\limits_{\Omega} \varphi_* \left(|\nabla f|\right) \, dm(x).$$

Finally, by absolute continuity of the indefinite integral and arbitrariness of ε and $\delta > 0$ we obtain (4.2).

Combining Proposition 2.1 and Corollary 4.1 we obtain the following statement.

Proposition 4.1. *Let $k = 2, \dots, n-1$, U be an open set in $\mathbb{R}^n$, $n \geqslant 3$, and let $f : U \to \mathbb{R}^m$, $m \geq 1$, be a continuous mapping in the class $W^{1,\varphi}_{\rm loc}(U)$ for some increasing function $\varphi : [0,\infty) \to [0,\infty)$, $\varphi(0) = 0$, such that (4.1) holds. Then, for every k-dimensional direction Γ for a.e. k-dimensional plane $\mathcal{P} \in \Gamma$, the restriction of the function f on the set $\mathcal{P} \cap U$ has the (N)-property (furthermore, it is locally absolutely continuous) with respect to the k-dimensional Hausdorff measure. Moreover, $H^k(f(E)) = 0$ whenever $\nabla_k f = 0$ on $E \subset P$ for a.e. $P \in \Gamma$.*

Here ∇_k denotes the k-dimensional gradient of the restriction of the mapping f to the k-dimensional plane P. However, the most important particular case of Proposition 4.1 for us is the following statement.

Theorem 4.2. *Let U be an open set in $\mathbb{R}^n$, $n \geqslant 3$, and let $\varphi : [0,\infty) \to [0,\infty)$ be an increasing function such that $\varphi(0) = 0$ and*

$$\int\limits_1^{\infty} \left[\frac{t}{\varphi(t)}\right]^{\frac{1}{n-2}} dt < \infty. \tag{4.5}$$

Then each continuous mapping $f : U \to \mathbb{R}^m$, $m \geq 1$, in the class $W^{1,\varphi}_{\rm loc}$ has the (N)-property (furthermore, it is locally absolutely continuous) with respect to the $(n-1)$-dimensional Hausdorff measure on a.e. hyperplane $\mathcal{P}$ which is parallel to a fixed hyperplane $\mathcal{P}_0$. Moreover, $H^{n-1}(f(E)) = 0$ whenever $|\nabla f| = 0$ on $E \subset \mathcal{P}$ for a.e. such $\mathcal{P}$.

Note that, if the condition (4.5) holds for an increasing function φ, then the function $\varphi_* = \varphi(ct)$ for $c > 0$ also satisfies (4.5). Moreover, the Hausdorff measures are quasi-invariant under quasi-isometries. By the Lindelöf property of $\mathbb{R}^n$, $U \setminus \{x_0\}$ can be covered by a countable collection of open segments of spherical rings in $U \setminus \{x_0\}$ centered at x_0 and each such segment can be mapped onto a rectangular oriented segment of $\mathbb{R}^n$ by some quasi-isometry, see, e.g., I.5.XI in [100]

for the Lindelöf theorem. Thus, applying Theorem 4.2 piecewise, we obtain the following conclusion.

Corollary 4.3. *Under (4.5) each $f \in W^{1,\varphi}_{\rm loc}$ has the (N)-property (furthermore, it is locally absolutely continuous) on a.e. sphere S centered at a prescribed point $x_0 \in \mathbb{R}^n$. Moreover, $H^{n-1}(f(E)) = 0$ whenever $|\nabla f| = 0$ on $E \subseteq S$ for a.e. such sphere S.*

Remark 4.2. In particular, (4.5) holds for the functions $\varphi(t) = t^p$, $p > n-1$, i.e., the given properties hold for $f \in W^{1,p}_{\rm loc}$, $p > n-1$.

Note also that (4.5) does not imply the (N)-property of $f : U \to \mathbb{R}^n$ in U with respect to the Lebesgue measure in $\mathbb{R}^n$. The latter conclusion follow, in particular, from the Ponomarev examples of homeomorphisms $f \in W^{1,p}_{\rm loc}(\mathbb{R}^n)$ for all $p < n$ without (N)-property of Lusin, see [133].

If $m < n-1$, then $H^{n-1}(\mathbb{R}^m) = 0$ and the (N)–property on a.e. hyperplane for the mapping f in Theorem 4.2 is obvious without the condition (4.5). However, the examples in the final section show that the condition (4.5) are not only sufficient but also necessary for the (N)-property on a.e. hyperplane if $m \geqslant n-1$, see Remark 13.2 and Theorem 13.1.

The connection of estimates of the Calderon type (2.2) with the (N)–property and differentiability was first found under the study of the so–called generalized Lipschitzians in the sense of Rado, see, e.g., [16] and V.3.6 in [135], and also the recent works [9], [70]and [141].

5 On BMO and FMO functions

The BMO space was introduced by John and Nirenberg in [67] and soon became one of the main concepts in harmonic analysis, complex analysis, partial differential equations and related areas, see, e.g., [51] and [143].

Let D be a domain in $\mathbb{R}^n$, $n \geqslant 1$. Recall that a real valued function $\varphi \in \mathrm{L}^1_{\rm loc}(D)$ is said to be of **bounded mean oscillation** in D, abbr.

$\varphi \in \mathrm{BMO(D)}$ or simply $\varphi \in \mathbf{BMO}$, if

$$\|\varphi\|_* = \sup_{B\subset D} \quad \fint_B |\varphi(z) - \varphi_B| \, dm(z) \ < \ \infty \tag{5.1}$$

where the supremum is taken over all balls B in D and

$$\varphi_B = \fint_B \varphi(z) \ dm(z) = \frac{1}{|B|} \int_B \varphi(z) \, dm(z) \tag{5.2}$$

is the mean value of the function φ over B. Note that $\mathrm{L}^\infty(\mathrm{D}) \subset \mathrm{BMO(D)} \subset \mathrm{L}^p_{\mathrm{loc}}(\mathrm{D})$ for all $1 \le p < \infty$, see, e.g., [143].

A function φ in BMO is said to have **vanishing mean oscillation**, abbr. $\varphi \in \mathbf{VMO}$, if the supremum in (5.1) taken over all balls B in D with $|B| < \varepsilon$ converges to 0 as $\varepsilon \to 0$. VMO has been introduced by Sarason in [157]. There are a number of papers devoted to the study of partial differential equations with coefficients of the class VMO, see, e.g., [19], [66], [116], [128] and [136].

Following [62], we say that a function $\varphi : D \to \mathbb{R}$ has **finite mean oscillation at a point** $z_0 \in D$, write $\varphi \in \mathrm{FMO}(x_0)$, if

$$\overline{\lim_{\varepsilon\to 0}} \ \ \fint_{B(z_0,\varepsilon)} |\varphi(z) - \tilde{\varphi}_\varepsilon(z_0)| \ dm(z) \ < \ \infty \tag{5.3}$$

where

$$\tilde{\varphi}_\varepsilon(z_0) = \fint_{B(z_0,\varepsilon)} \varphi(z) \ dm(z) \tag{5.4}$$

is the mean value of the function $\varphi(z)$ over the ball $B(z_0, \varepsilon)$. The condition (5.3) includes the assumption that φ is integrable in some neighborhood of the point z_0. We also say that a function φ is of **finite mean oscillation in the domain** D, write $\varphi \in \mathrm{FMO}(D)$ or simply $\varphi \in \mathbf{FMO}$, if this property holds at every point $x_0 \in D$.

Proposition 5.1. If for some collection of numbers $\varphi_\varepsilon \in \mathbb{R}$, $\varepsilon \in (0, \varepsilon_0]$,

$$\overline{\lim_{\varepsilon\to 0}} \ \ \fint_{B(z_0,\varepsilon)} |\varphi(z) - \varphi_\varepsilon| \ dm(z) \ < \infty \,, \tag{5.5}$$

then φ is of finite mean oscillation at z_0.

Indeed, by the triangle inequality

$$\fint_{B(x_0,\varepsilon)} |\varphi(x)-\overline{\varphi}_\varepsilon|\, dm(x) \ \leq \fint_{B(x_0,\varepsilon)} |\varphi(x)-\varphi_\varepsilon|\, dm(x) \,+\, |\varphi_\varepsilon-\overline{\varphi}_\varepsilon|$$

$$\leq\ 2\cdot \fint_{B(x_0,\varepsilon)} |\varphi(x)-\varphi_\varepsilon|\, dm(x)\ .$$

Choosing in Proposition 5.1 $\varphi_\varepsilon \equiv 0$, $\varepsilon \in (0,\varepsilon_0]$, we obtain the following statement.

Corollary 5.1. If for a point $z_0 \in D$

$$\overline{\lim_{\varepsilon\to 0}} \fint_{B(z_0,\varepsilon)} |\varphi(z)|\, dm(z)\ <\ \infty\ , \tag{5.6}$$

then φ has finite mean oscillation at z_0.

Recall that a point $z_0 \in D$ is called a Lebesgue point of a function $\varphi : D \to \mathbb{R}$ if φ is integrable in a neighborhood of z_0 and

$$\lim_{\varepsilon\to 0} \fint_{B(z_0,\varepsilon)} |\varphi(z)-\varphi(z_0)|\, dm(z)\ =\ 0\ . \tag{5.7}$$

It is known that almost every point in D is a Lebesgue point for every function $\varphi \in L^1(D)$. We thus have the following corollary.

Corollary 5.2. Every function $\varphi : D \to \mathbb{R}$, which is locally integrable, has a finite mean oscillation at almost every point in D.

Remark 5.1. Note that the function $\varphi(z) = \log(1/|z|)$ belongs to BMO in the unit disk Δ, see, e.g., [143], p. 5, and hence also to FMO. However, $\tilde{\varphi}_\varepsilon(0) \to \infty$ as $\varepsilon \to 0$, showing that the condition (5.6) is only sufficient but not necessary for a function φ to be of finite mean oscillation at z_0.

Clearly that BMO $\subset$ FMO. By definition FMO $\subset L^1_{\rm loc}$ but FMO is not a subset of $L^p_{\rm loc}$ for any $p > 1$ in comparison with $\mathrm{BMO}_{\rm loc} \subset L^p_{\rm loc}$ for all $p \in [1,\infty)$. Here $\mathrm{BMO}_{\rm loc}$ stands for the local version of

the class BMO. So, let us give examples showing that FMO is not $\mathrm{BMO}_{\mathrm{loc}}$.

Example 1. Set $z_n = 2^{-n}$, $r_n = 2^{-pn^2}$, $p > 1$, $c_n = 2^{2n^2}$, $D_n = \{z \in \mathbb{C} : |z - z_n| < r_n\}$, and

$$\varphi(z) = \sum_{n=1}^{\infty} c_n \chi(D_n).$$

Evidently by Corollary 5.1 that $\varphi \in FMO(\mathbb{C} \setminus \{0\})$.

To prove that $\varphi \in FMO(0)$, fix N such that $(p-1)N > 1$, and set $\varepsilon = \varepsilon_N = z_N + r_N$. Then $\bigcup\limits_{n \geq N} D_n \subset \mathbb{D}(\varepsilon) := \{z \in \mathbb{C} : |z| < \varepsilon\}$ and

$$\begin{aligned}
\int\limits_{\mathbb{D}(\varepsilon)} \varphi \;&=\; \sum_{n \geq N} \int\limits_{D_n} \varphi \;=\; \pi \sum_{n \geq N} c_n r_n^2 \\
&=\; \sum_{n \geq N} 2^{2(1-p)n^2} \;<\; \sum_{n \geq N} 2^{2(1-p)n} \\
&<\; C \cdot [2^{(1-p)N}]^2 \;<\; 2C\varepsilon^2 \,.
\end{aligned}$$

Hence $\varphi \in FMO(0)$ and, consequently, $\varphi \in FMO(\mathbb{C})$.

On the other hand

$$\int\limits_{\mathbb{D}(\varepsilon)} \varphi^p \;=\; \pi \sum_{n > N} c_n^p \cdot r_n^2 \;=\; \sum_{n > N} 1 \;=\; \infty.$$

Hence $\varphi \notin L^p(\mathbb{D}(\varepsilon))$ and therefore $\varphi \notin BMO_{\mathrm{loc}}$ because $\mathrm{BMO}_{\mathrm{loc}} \subset L^p_{\mathrm{loc}}$ for all $p \in [1, \infty)$.

Example 2. We conclude this section by constructing functions $\varphi : \mathbb{C} \to \mathbb{R}$ of the class $C^\infty(\mathbb{C} \setminus \{0\})$ which belongs to FMO but not to L^p_{loc} for any $p > 1$ and hence not to $\mathrm{BMO}_{\mathrm{loc}}$.

In this example, $p = 1 + \delta$ with an arbitrarily small $\delta > 0$. Set

$$\varphi_0(z) = \begin{cases} e^{\frac{1}{|z|^2 - 1}}, & \text{if } |z| < 1, \\ 0, & \text{if } |z| \geq 1. \end{cases} \tag{5.8}$$

Then φ_0 belongs to C_0^∞ and hence to $BMO_{\rm loc}$. Consider the function

$$\varphi(z) = \begin{cases} \varphi_k(z), & \text{if } z \in B_k, \\ 0, & \text{if } z \in \mathbb{C} \setminus \bigcup B_k \end{cases} \tag{5.9}$$

where $B_k = B(z_k, r_k)$, $z_k = 2^{-k}$, $r_k = 2^{-(1+\delta)k^2}$, $\delta > 0$, and

$$\varphi_k(z) = 2^{2k^2} \varphi_0 \left(\frac{z - z_k}{r_k} \right), \quad z \in B_k, \quad k = 2, 3, \dots . \tag{5.10}$$

Then φ is smooth in $\mathbb{C} \setminus \{0\}$ and, thus, belongs to $\mathrm{BMO}_{\rm loc}(\mathbb{C} \setminus \{0\})$, and hence to $\mathrm{FMO}(\mathbb{C} \setminus \{0\})$.

Now,

$$\int\limits_{B_k} \varphi_k(z)\, dm(z) \; = \; 2^{-2\delta k^2} \int\limits_{\mathbb{C}} \varphi_0(z)\, dm(z) \; . \tag{5.11}$$

Setting

$$K = K(\varepsilon) = \left[\log_2 \frac{1}{\varepsilon} \right] \le \log_2 \frac{1}{\varepsilon} \; , \tag{5.12}$$

where $[A]$ denotes the integral part of the number A, we have

$$J = \fint_{D(\varepsilon)} \varphi(z)\, dm(z) \le I \cdot \sum_{k=K}^{\infty} 2^{-2\delta k^2} / \pi 2^{-2(K+1)}, \tag{5.13}$$

where $I = \int\limits_{\mathbb{C}} \varphi(z)\, dm(z)$. If $K\delta > 1$, i.e. $K > 1/\delta$, then

$$\sum_{k=K}^{\infty} 2^{-2\delta k^2} \le \sum_{k=K}^{\infty} 2^{-2k} = 2^{-2K} \sum_{k=0}^{\infty} \left(\frac{1}{4} \right)^k = \frac{4}{3} \cdot 2^{-2K}, \tag{5.14}$$

i.e., $J \le 16I/3\pi$. Hence

$$\overline{\lim_{\varepsilon \to 0}} \; \fint_{B(\varepsilon)} \varphi(z)\, dm(z) \; < \; \infty \; . \tag{5.15}$$

Thus, $\varphi \in \mathrm{FMO}$ by Corollary 5.1.

On the other hand,

$$\int\limits_{B_k} \varphi_k^{1+\delta}(z)\, dm(z) = \int\limits_{\mathbb{C}} \varphi_0^{1+\delta}(z)\, dm(z) \tag{5.16}$$

and hence $\varphi \notin L^{1+\delta}(U)$ for any neighborhood U of 0.

6 On some integral conditions

For every non-decreasing function $\Phi : [0,\infty] \to [0,\infty]$, the **inverse function** $\Phi^{-1} : [0,\infty] \to [0,\infty]$ can be well defined by setting

$$\Phi^{-1}(\tau) \;=\; \inf_{\Phi(t)\geq\tau} t\ . \tag{6.1}$$

As usual, here inf is equal to ∞ if the set of $t \in [0,\infty]$ such that $\Phi(t) \geq \tau$ is empty. Note that the function Φ^{-1} is non-decreasing, too.

Remark 6.1. Immediately by the definition it is evident that

$$\Phi^{-1}(\Phi(t)) \;\leq\; t \qquad\qquad \forall\, t \in [0,\infty] \tag{6.2}$$

with the equality in (6.2) except intervals of constancy of the function $\Phi(t)$.

Since the mapping $t \mapsto t^p$ for every positive p is a sense–preserving homeomorphism $[0,\infty]$ onto $[0,\infty]$ we may write Theorem 2.1 from [154] in the following form which is more convenient for further applications. Here, in (6.4) and (6.5), we complete the definition of integrals by ∞ if $\Phi_p(t) = \infty$, correspondingly, $H_p(t) = \infty$, for all $t \geq T \in [0,\infty)$. The integral in (6.5) is understood as the Lebesgue-Stieltjes integral and the integrals in (6.4) and (6.6)–(6.9) as the ordinary Lebesgue integrals.

Proposition 6.1. *Let $\Phi : [0,\infty] \to [0,\infty]$ be a non-decreasing function. Set*

$$H_p(t) \;=\; \log \Phi_p(t)\ , \qquad \Phi_p(t) = \Phi\left(t^p\right)\ , \quad p \in (0,\infty)\,. \tag{6.3}$$

Then the equality

$$\int\limits_{\delta}^{\infty} H_p'(t)\,\frac{dt}{t} \;=\; \infty \tag{6.4}$$

implies the equality

$$\int\limits_{\delta}^{\infty} \frac{dH_p(t)}{t} \;=\; \infty \tag{6.5}$$

and (6.5) is equivalent to

$$\int\limits_{\delta}^{\infty} H_p(t)\,\frac{dt}{t^2} \;=\; \infty \tag{6.6}$$

for some $\delta > 0$, *and (6.6) is equivalent to every of the equalities:*

$$\int\limits_{0}^{\Delta} H_p\left(\frac{1}{t}\right)\; dt \;=\; \infty \tag{6.7}$$

for some $\Delta > 0$,

$$\int\limits_{\delta_*}^{\infty} \frac{d\eta}{H_p^{-1}(\eta)} \;=\; \infty \tag{6.8}$$

for some $\delta_* > H(+0)$,

$$\int\limits_{\delta_*}^{\infty} \frac{d\tau}{\tau\Phi_p^{-1}(\tau)} \;=\; \infty \tag{6.9}$$

for some $\delta_* > \Phi(+0)$.

Moreover, (6.4) is equivalent to (6.5) and hence (6.4)–(6.9) are equivalent each to other if Φ *is in addition absolutely continuous. In particular, all the conditions (6.4)–(6.9) are equivalent if* Φ *is convex and non–decreasing.*

It is easy to see that conditions (6.4)–(6.9) become weaker as p increases, see e.g. (6.6). It is necessary to give one more explanation. From the right hand sides in the conditions (6.4)–(6.9) we have in mind $+\infty$. If $\Phi_p(t) = 0$ for $t \in [0, t_*]$, then $H_p(t) = -\infty$ for $t \in [0, t_*]$ and we complete the definition $H_p'(t) = 0$ for $t \in [0, t_*]$. Note, the conditions (6.5) and (6.6) exclude that t_* belongs to the interval of integrability because in the contrary case the left hand sides in (6.5) and (6.6) are either equal to $-\infty$ or indeterminate. Hence we may assume in (6.4)–(6.7) that $\delta > t_0$, correspondingly, $\Delta < 1/t_0$ where $t_0 \colon = \sup\limits_{\Phi_p(t)=0} t$, $t_0 = 0$ if $\Phi_p(0) > 0$.

Recall that a function $\Phi : [0,\infty] \to [0,\infty]$ is called **convex** if

$$\Phi(\lambda t_1 + (1-\lambda) t_2) \ \leq \ \lambda\, \Phi(t_1) \ + \ (1-\lambda)\, \Phi(t_2)$$

for all t_1 and $t_2 \in [0,\infty]$ and $\lambda \in [0,1]$.

Lemma 3.1 from [154] can be written in the following form.

Lemma 6.1. *Let $Q : \mathbb{B}^n \to [0,\infty]$ be a measurable function and let $\Phi : [0,\infty] \to (0,\infty]$ be a non-decreasing convex function. Then*

$$\int\limits_0^1 \frac{dr}{rq^{\frac{1}{p}}(r)} \ \geq \ \frac{1}{n} \int\limits_{eM}^{\infty} \frac{d\tau}{\tau\, [\Phi^{-1}(\tau)]^{\frac{1}{p}}} \qquad \forall \quad p \in (0,\infty) \tag{6.10}$$

where $q(r)$ is the average of the function $Q(x)$ over the sphere $|x| = r$ and M is the average of the function $\Phi \circ Q$ over the unit ball $\mathbb{B}^n$.

Remark 6.2. Note that (6.10) is equivalent for each $p \in (0,\infty)$ to the inequality

$$\int\limits_0^1 \frac{dr}{rq^{\frac{1}{p}}(r)} \ \geq \ \frac{1}{n} \int\limits_{eM}^{\infty} \frac{d\tau}{\tau \Phi_p^{-1}(\tau)}\,, \qquad \Phi_p(t) \ : = \ \Phi(t^p)\,. \tag{6.11}$$

Theorem 6.1. *Let $Q : \mathbb{B}^n \to [0,\infty]$ be a measurable function such that*

$$\int\limits_{\mathbb{B}^n} \Phi(Q(x))\, dm(x) \ < \ \infty \tag{6.12}$$

where $\Phi : [0,\infty] \to [0,\infty]$ is a non-decreasing convex function such that

$$\int\limits_{\delta_0}^{\infty} \frac{d\tau}{\tau\, [\Phi^{-1}(\tau)]^{\frac{1}{p}}} \ = \ \infty\,, \qquad p \in (0,\infty)\,, \tag{6.13}$$

for some $\delta_0 \ > \Phi(0)$. Then

$$\int\limits_0^1 \frac{dr}{rq^{\frac{1}{p}}(r)} \ = \ \infty \tag{6.14}$$

where $q(r)$ is the average of $Q(x)$ over the sphere $|x| = r$.

Remark 6.3. In view of Proposition 6.1, if (6.12) holds, then each of the conditions (6.4)–(6.9) implies the condition (6.14).

7 Moduli of families of surfaces

Let ω be an open set in $\overline{\mathbb{R}^k}$, $k = 1, \ldots, n-1$. A (continuous) mapping $S : \omega \to \mathbb{R}^n$ is called a k-dimensional surface S in $\mathbb{R}^n$. Sometimes we call the image $S(\omega) \subseteq \mathbb{R}^n$ the surface S, too. The number of preimages

$$N(S, y) = \operatorname{card} S^{-1}(y) = \operatorname{card}\{x \in \omega : \ S(x) = y\}, \ \ y \in \mathbb{R}^n \quad (7.1)$$

is said to be a **multiplicity function** of the surface S. In other words, $N(S, y)$ denotes the multiplicity of covering of the point y by the surface S. It is known that the multiplicity function is lower semicontinuous, i.e.,

$$N(S, y) \ \geqslant \ \liminf_{m\to\infty} \ N(S, y_m)$$

for every sequence $y_m \in \mathbb{R}^n$, $m = 1, 2, \ldots$, such that $y_m \to y \in \mathbb{R}^n$ as $m \to \infty$; see, e.g., [135], p. 160. Thus, the function $N(S, y)$ is Borel measurable and hence measurable with respect to every Hausdorff measure H^k; see, e.g., [156], p. 52.

Recall that a k-dimensional Hausdorff area in $\mathbb{R}^n$ (or simply **area**) associated with a surface $S : \omega \to \mathbb{R}^n$ is given by

$$\mathcal{A}_S(B) \ = \ \mathcal{A}^k_S(B) \ := \int\limits_B N(S, y)\, dH^k y \qquad (7.2)$$

for every Borel set $B \subseteq \mathbb{R}^n$ and, more generally, for an arbitrary set that is measurable with respect to H^k in $\mathbb{R}^n$, cf. 3.2.1 in [31]. The surface S is called **rectifiable** if $\mathcal{A}_S(\mathbb{R}^n) < \infty$, see 9.2 in [115].

If $\varrho : \mathbb{R}^n \to [0, \infty]$ is a Borel function, then its **integral over** S is defined by the equality

$$\int\limits_S \varrho \, d\mathcal{A} \ := \int\limits_{\mathbb{R}^n} \varrho(y)\, N(S, y)\, dH^k y\, . \qquad (7.3)$$

Given a family Γ of k-dimensional surfaces S, a Borel function $\varrho : \mathbb{R}^n \to [0,\infty]$ is called **admissible** for Γ, abbr. $\varrho \in \operatorname{adm}\Gamma$, if

$$\int\limits_S \varrho^k \, d\mathcal{A} \ \geqslant\ 1 \tag{7.4}$$

for every $S \in \Gamma$. Given $p \in (0,\infty)$, the **p-modulus** of Γ is the quantity

$$M_p(\Gamma) \ =\ \inf_{\varrho\in\operatorname{adm}\Gamma} \int\limits_{\mathbb{R}^n} \varrho^p(x)\, dm(x). \tag{7.5}$$

We also set

$$M(\Gamma) \ =\ M_n(\Gamma) \tag{7.6}$$

and call the quantity $M(\Gamma)$ the **modulus of the family** Γ. The modulus is itself an outer measure on the collection of all families Γ of k-dimensional surfaces.

We say that Γ_2 is **minorized** by Γ_1 and write $\Gamma_2 > \Gamma_1$ if every $S \subset \Gamma_2$ has a subsurface that belongs to Γ_1. It is known that $M_p(\Gamma_1) \geqslant M_p(\Gamma_2)$, see [32], p. 176-178. We also say that a property P holds for p**-a.e.** (almost every) k-dimensional surface S in a family Γ if a subfamily of all surfaces of Γ, for which P fails, has the p-modulus zero. If $0 < q < p$, then P also holds for q-a.e. S, see Theorem 3 in [32]. In the case $p = n$, we write simply a.e.

Remark 7.1. The definition of the modulus immediately implies that, for every $p \in (0,\infty)$ and $k = 1,\ldots,n-1$

(1) p-a.e. k-dimensional surface in $\mathbb{R}^n$ is rectifiable,

(2) given a Borel set B in $\mathbb{R}^n$ of (Lebesgue) measure zero,

$$\mathcal{A}_S(B) \ =\ 0 \tag{7.7}$$

for p-a.e. k-dimensional surface S in $\mathbb{R}^n$.

The following lemma was first proved in [81], see also Lemma 9.1 in [115].

Lemma 7.1. *Let $k = 1,\ldots,n-1$, $p \in [k,\infty)$, and let C be an open cube in $\mathbb{R}^n$, $n \geqslant 2$, whose edges are parallel to coordinate axis.*

If a property P holds for p-a.e. k-dimensional surface S in C, then P also holds for a.e. k-dimensional plane in C that is parallel to a k-dimensional coordinate plane H.

The latter a.e. is related to the Lebesgue measure in the corresponding $(n-k)$-dimensional coordinate plane $H^{\perp}$ that is perpendicular to H.

The following statement, see Theorem 2.11 in [82] or Theorem 9.1 in [115], is an analogue of the Fubini theorem, cf., e.g., [156], p. 77. It extends Theorem 33.1 in [173], cf. also Theorem 3 in [32], Lemma 2.13 in [112], and Lemma 8.1 in [115].

Theorem 7.1. *Let $k = 1, \dots, n-1$, $p \in [k, \infty)$, and let E be a subset in an open set $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$. Then E is measurable by Lebesgue in $\mathbb{R}^n$ if and only if E is measurable with respect to area on p-a.e. k-dimensional surface S in Ω. Moreover, $|E| = 0$ if and only if*

$$\mathcal{A}_S(E) \;=\; 0 \tag{7.8}$$

on p-a.e. k-dimensional surface S in Ω.

Remark 7.2. Say by the Lusin theorem, see e.g. Section 2.3.5 in [31], for every measurable function $\varrho : \mathbb{R}^n \to [0, \infty]$, there is a Borel function $\varrho^* : \mathbb{R}^n \to [0, \infty]$ such that $\varrho^* = \varrho$ a.e. in $\mathbb{R}^n$. Thus, by Theorem 7.1, ϱ is measurable on p-a.e. k-dimensional surface S in $\mathbb{R}^n$ for every $p \in (0, \infty)$ and $k = 1, \dots, n-1$.

We say that a Lebesgue measurable function $\varrho : \mathbb{R}^n \to [0, \infty]$ is ***p*-extensively admissible** for a family Γ of k-dimensional surfaces S in $\mathbb{R}^n$, abbr. $\varrho \in \operatorname{ext}_p \operatorname{adm} \Gamma$, if

$$\int\limits_S \varrho^k \, d\mathcal{A} \;\geqslant\; 1 \tag{7.9}$$

for p-a.e. $S \in \Gamma$. The ***p*-extensive modulus** $\overline{M}_p(\Gamma)$ of Γ is the quantity

$$\overline{M}_p(\Gamma) \;=\; \inf \int\limits_{\mathbb{R}^n} \varrho^p(x) \, dm(x) \tag{7.10}$$

where the infimum is taken over all $\varrho \in \mathrm{ext}_p \, \mathrm{adm}\, \Gamma$. In the case $p = n$, we use the notations $\overline{M}(\Gamma)$ and $\varrho \in \mathrm{ext}\, \mathrm{adm}\, \Gamma$, respectively. For every $p \in (0, \infty)$, $k = 1, \ldots, n-1$, and every family Γ of k-dimensional surfaces in $\mathbb{R}^n$,

$$\overline{M}_p(\Gamma) \;=\; M_p(\Gamma)\,. \tag{7.11}$$

8 Lower and ring Q-homeomorphisms

The following concept is motivated by Gehring's ring definition of quasiconformality in [34].

Given domains D and D' in $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$, $n \geqslant 2$, $x_0 \in \overline{D} \setminus \{\infty\}$, and a measurable function $Q : D \to (0, \infty)$, we say that a homeomorphism $f : D \to D'$ is a **lower Q-homeomorphism at the point x_0** if

$$M(f\Sigma_\varepsilon) \;\geqslant\; \inf_{\varrho \in \mathrm{ext}\, \mathrm{adm}\, \Sigma_\varepsilon} \int\limits_{D \cap R_\varepsilon} \frac{\varrho^n(x)}{Q(x)}\, dm(x) \tag{8.1}$$

for every ring

$$R_\varepsilon \;=\; \{x \in \mathbb{R}^n : \varepsilon < |\, x - x_0| < \varepsilon_0\}\,, \qquad \varepsilon \in (0, \varepsilon_0)\,,\ \varepsilon_0 \in (0, d_0)\,,$$

where

$$d_0 \;=\; \sup_{x \in D} |x - x_0|\,, \tag{8.2}$$

and Σ_ε denotes the family of all intersections of the spheres

$$S(x_0, r) \;=\; \{x \in \mathbb{R}^n : |\, x - x_0| = r\}\,, \qquad r \in (\varepsilon, \varepsilon_0)\,,$$

with D. As usual, the notion can be extended to the case $x_0 = \infty \in \overline{D}$ by applying the inversion T with respect to the unit sphere in $\overline{\mathbb{R}^n}$, $T(x) = x/|\, x|^2$, $T(\infty) = 0$, $T(0) = \infty$. Namely, a homeomorphism $f : D \to D'$ is a **lower Q-homeomorphism at** $\infty \in \overline{D}$ if $F = f \circ T$ is a lower Q_*-homeomorphism with $Q_* = Q \circ T$ at 0.

We also say that a homeomorphism $f : D \to \overline{\mathbb{R}^n}$ is a **lower Q-homeomorphism in D** if f is a lower Q-homeomorphism at every point $x_0 \in \overline{D}$.

Recall the criterion for homeomorphisms in $\mathbb{R}^n$ to be lower Q-homeomorphisms, see Theorem 2.1 in [81] or Theorem 9.2 in [115].

Proposition 8.1. *Let D and D' be domains in $\overline{\mathbb{R}^n}$, $n \geqslant 2$, let $x_0 \in \overline{D} \setminus \{\infty\}$, and let $Q : D \to (0, \infty)$ a measurable function. A homeomorphism $f : D \to D'$ is a lower Q-homeomorphism at x_0 if and only if*

$$M(f\Sigma_\varepsilon) \;\geqslant\; \int\limits_{\varepsilon}^{\varepsilon_0} \frac{dr}{||\,Q||\,_{n-1}(r)} \quad \forall\, \varepsilon \in (0, \varepsilon_0)\,,\ \varepsilon_0 \in (0, d_0)\,, \tag{8.3}$$

where

$$||Q||\,_{n-1}(r) = \left(\int\limits_{D(x_0,r)} Q^{n-1}(x)\; d\mathcal{A} \right)^{\frac{1}{n-1}} \tag{8.4}$$

is the L_{n-1}-norm of Q over $D(x_0, r) = \{x \in D : |x - x_0| = r\} = D \cap S(x_0, r)$.

Note that the infimum of expression from the right-hand side in (8.1) is attained for the function

$$\varrho_0(x) = \frac{Q(x)}{||Q||\,_{n-1}(|x|)}\,.$$

Now, given a domain D and two sets E and F in $\overline{\mathbb{R}^n}$, $n \geqslant 2$, $\Delta(E, F, D)$ denotes the family of all paths $\gamma : [a, b] \to \overline{\mathbb{R}^n}$ that join E and F in D, i.e., $\gamma(a) \in E$, $\gamma(b) \in F$, and $\gamma(t) \in D$ for $a < t < b$. Set

$$A(r_1, r_2, x_0) \;=\; \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}, \tag{8.5}$$

$$S(x_0, r_i) \;=\; \{x \in \mathbb{R}^n : |x - x_0| = r_i\}, \quad i = 1, 2. \tag{8.6}$$

Given domains D in $\mathbb{R}^n$ and D' in $\overline{\mathbb{R}^n}$, $n \geqslant 2$, and a measurable function $Q : D \to [0, \infty]$, they say that a homeomorphism $f : D \to D'$ is a **ring Q-homeomorphism at a point** $x_0 \in D$ if

$$M(\Delta(fS_1, fS_2, fD)) \;\leqslant \int\limits_{A} Q(x) \cdot \eta^n(|x - x_0|)\; dm(x) \tag{8.7}$$

for every ring $A = A(r_1, r_2, x_0)$, $0 < r_1 < r_2 < d_0 = \mathrm{dist}(x_0, \partial D)$, and for every measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)\, dr \ = \ 1\,. \tag{8.8}$$

The notion was first introduced in the work [154] in the connection with investigations of the Beltrami equations in the plane and then it was extended to the space in the work [152].

Let us recall the following criterion for ring Q-homeomorphisms, see Theorem 3.15 in [152] or Theorem 7.2 in [115].

Proposition 8.2. *Let D be a domain in $\mathbb{R}^n$, $n \geqslant 2$, and $Q : D \to [0, \infty]$ a measurable function. A homeomorphism $f : D \to \mathbb{R}^n$ is a ring Q-homeomorphism at a point $x_0 \in D$ if and only if for every $0 < r_1 < r_2 < d_0 = \mathrm{dist}\,(x_0, \partial D)$*

$$M(\Delta(fS_1, fS_2, fD)) \ \leqslant \ \frac{\omega_{n-1}}{I^{n-1}} \tag{8.9}$$

where ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$, $S_j = \{x \in \mathbb{R}^n : |x - x_0| = r_j\}$, $j = 1, 2$, and

$$I \ = \ I(r_1, r_2) \ = \ \int\limits_{r_1}^{r_2} \frac{dr}{r q_{x_0}^{\frac{1}{n-1}}(r)}$$

and $q_{x_0}(r)$ is the mean value of $Q(x)$ over the sphere $|x - x_0| = r$.

Note that the infimum from the right-hand side in (8.7) holds for the function

$$\eta_0(r) = \frac{1}{I r q_{x_0}^{\frac{1}{n-1}}(r)}\,.$$

Remark 8.1. By the Hesse and Ziemer equalities in [58] and [182], see also the appendixes A3 and A6 in [115], we have

$$M(\Delta(fS_1, fS_2, fD)) \ \leqslant \ \frac{1}{M^{n-1}(f\Sigma)} \tag{8.10}$$

because $f\Sigma \subset \Sigma(fS_1, fS_2, fD)$ where Σ is a collection of all spheres centered at x_0 between S_1 and S_2 and $\Sigma(fS_1, fS_2, fD)$ consists of all $(n-1)$-dimensional surfaces in fD that separate fS_1 and fS_2.

Thus, comparing the above criteria for lower and ring Q-homeomorphisms, we obtain the following conclusion at inner points.

Corollary 8.1. *Each lower Q-homeomorphism $f : D \to D'$ in $\mathbb{R}^n$, $n \geqslant 2$, at a point $x_0 \in D$ is a ring Q^*-homeomorphism with $Q^* = Q^{n-1}$ at the point x_0.*

Corollary 8.2. *Each lower Q-homeomorphism $f : D \to D'$ in the plane at a point $x_0 \in D$ is a ring Q-homeomorphism at the point x_0.*

It was proved in the work [84] that each homeomorphism f of finite distortion in the plane is a lower Q-homeomorphism with $Q(x) = K_f(x)$. In the next section we show that the same is true for a homeomorphism f of finite distortion in $\mathbb{R}^n$, $n \geqslant 3$, if, in addition, $f \in W^{1,\varphi}_{\rm loc}$ where φ satisfies the Calderon type condition (4.5).

9 Lower Q-homeomorphisms and Orlicz-Sobolev classes

The following statement is key for our further research.

Theorem 9.1. *Let D and D' be domains in $\mathbb{R}^n$, $n \geqslant 3$, and let $\varphi : [0, \infty) \to [0, \infty)$ be an increasing function such that $\varphi(0) = 0$ and*

$$\int\limits_1^{\infty} \left[\frac{t}{\varphi(t)}\right]^{\frac{1}{n-2}} dt < \infty. \tag{9.1}$$

Then each homeomorphism $f : D \to D'$ of finite distortion in the class $W^{1,\varphi}_{\rm loc}$ is a lower Q-homeomorphism at every point $x_0 \in \overline{D}$ with $Q(x) = K_f(x)$.

Proof. Let B be a (Borel) set of all points $x \in D$ where f has a total differential $f'(x)$ and $J_f(x) \neq 0$. Then, applying Kirszbraun's theorem and uniqueness of approximate differential, see, e.g., 2.10.43

and 3.1.2 in [31], we see that B is the union of a countable collection of Borel sets B_l, $l = 1, 2, \ldots$, such that $f_l = f|_{B_l}$ are bi-Lipschitz homeomorphisms, see, e.g., 3.2.2 as well as 3.1.4 and 3.1.8 in [31]. With no loss of generality, we may assume that the B_l are mutually disjoint. Denote also by B_* the rest of all points $x \in D$ where f has the total differential but with $f'(x) = 0$.

By the construction the set $B_0 := D \setminus (B \bigcup B_*)$ has Lebesgue measure zero, see Theorem 3.1. Hence by Theorem 7.1 $\mathcal{A}_S(B_0) = 0$ for a.e. hypersurface S in $\mathbb{R}^n$ and, in particular, for a.e. sphere $S_r := S(x_0, r)$ centered at a prescribed point $x_0 \in \overline{D}$. Thus, by Corollary 4.3 $\mathcal{A}_{S_r^*}(f(B_0)) = 0$ as well as $\mathcal{A}_{S_r^*}(f(B_*)) = 0$ for a.e. S_r where $S_r^* = f(S_r)$.

Let Γ be the family of all intersections of the spheres S_r, $r \in (\varepsilon, \varepsilon_0)$, $\varepsilon_0 < d_0 = \sup\limits_{x \in D} |x - x_0|$, with the domain D. Given $\varrho_* \in adm\, f(\Gamma)$, $\varrho_* \equiv 0$ outside $f(D)$, set $\varrho \equiv 0$ outside D and on B_0

$$\varrho(x) \ : \ = \ \varrho_*(f(x))\|f'(x)\| \qquad \text{for } x \in D \setminus B_0 \, .$$

Arguing piecewise on B_l, $l = 1, 2, \ldots$, we have by 1.7.6 and 3.2.2 in [31] that

$$\int\limits_{S_r} \varrho^{n-1} \, d\mathcal{A} \ \geqslant \ \int\limits_{S_*^r} \varrho_*^{n-1} \, d\mathcal{A} \ \geqslant \ 1$$

for a.e. S_r and, thus, $\varrho \in \operatorname{ext} \operatorname{adm} \Gamma$.

The change of variables on each B_l, $l = 1, 2, \ldots$, see, e.g., Theorem 3.2.5 in [31], and countable additivity of integrals give the estimate

$$\int\limits_{D} \frac{\varrho^n(x)}{K_f(x)} \, dm(x) \ \leqslant \ \int\limits_{f(D)} \varrho_*^n(x) \, dm(x)$$

and the proof is complete.

Corollary 9.1. *Each homeomorphism f of finite distortion in $\mathbb{R}^n$, $n \geqslant 3$, in the class $W^{1,p}_{\mathrm{loc}}$ for $p > n - 1$ is a lower Q-homeomorphism at every point $x_0 \in \overline{D}$ with $Q(x) = K_f(x)$.*

Corollary 9.2. *In particular, each homeomorphism f of finite distortion in $\mathbb{R}^n$, $n \geqslant 3$, with $K_f \in L^p_{\rm loc}$ for $p > n-1$ is a lower Q-homeomorphism at every point $x_0 \in \overline{D}$ with $Q(x) = K_f(x)$.*

Corollary 9.3. *Under the hypotheses of Theorem 9.1, each homeomorphism of finite distortion $f \in W^{1,\varphi}_{\rm loc}$, in particular, $f \in W^{1,p}_{\rm loc}$ for $p > n-1$, is a ring Q_*-homeomorphism at every inner point $x_0 \in D$ with $Q_*(x) = [K_f(x)]^{n-1}$.*

10 Equicontinuous and normal families

First of all, recall some general facts on normal families of mappings in metric spaces. Let (X, d) and (X', d') be metric spaces with distances d and d', respectively. A family $\mathfrak{F}$ of continuous mappings $f : X \to X'$ is said to be **normal** if every sequence of mappings $f_m \in \mathfrak{F}$ has a subsequence f_{m_k} converging uniformly on each compact set $C \subset X$ to a continuous mapping. Normality is closely related to the following. A family $\mathfrak{F}$ of mappings $f : X \to X'$ is said to be **equicontinuous at a point** $x_0 \in X$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that $d'(f(x), f(x_0)) < \varepsilon$ for all $f \in \mathfrak{F}$ and $x \in X$ with $d(x, x_0) < \delta$. The family $\mathfrak{F}$ is called **equicontinuous** if $\mathfrak{F}$ is equicontinuous at every point $x_0 \in X$.

Given a domain G in $\mathbb{R}^n$, $n \geqslant 2$, and an increasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$, $M \in [0, \infty)$ and $x_0 \in G$, denote by $\mathfrak{F}^{\varphi}_M$ the collection of all continuous mappings $f : G \to \mathbb{R}^m$, $m \geqslant 1$, in the class $W^{1,1}_{\rm loc}$ such that $f(x_0) = 0$ and

$$\int\limits_G \varphi\left(|\nabla f|\right)\, dm(x) \leqslant M\ . \tag{10.1}$$

By Proposition 2.3 and Remark 2.1 and the Arzela-Ascoli theorem we obtain the following statement, cf., e.g., Theorem 8.1 in [64] and Theorem 4.3 in [45].

Corollary 10.1. If the function φ satisfies the condition

$$\int\limits_1^{\infty} \left[\frac{t}{\varphi(t)}\right]^{\frac{1}{n-1}} dt < \infty\,, \tag{10.2}$$

then the class $\mathfrak{F}^{\varphi}_M$ is equicontinuous and hence normal. If in addition φ is convex, then the class $\mathfrak{F}^{\varphi}_M$ is also closed with respect to the locally uniform convergence.

Further we give the corresponding theorems for the classes of homeomorphic mappings under the condition (10.4) which is weaker than (10.2) and without (locally) uniform constraints of the type (10.1) in these classes.

In what follows, we use in $\overline{\mathbb{R}^n} = \mathbb{R}^n \bigcup \{\infty\}$ the **spherical (chordal) metric** $h(x,y) = |\pi(x) - \pi(y)|$ where π is the stereographic projection of $\overline{\mathbb{R}^n}$ onto the sphere $S^n(\frac{1}{2}e_{n+1}, \frac{1}{2})$ in $\mathbb{R}^{n+1}$:

$$h(x,y) = \frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}\,,\ x \neq \infty \neq y,\ \ h(x,\infty) = \frac{1}{\sqrt{1+|x|^2}}\,.$$

Thus, by definition $h(x,y) \leqslant 1$ for all x and $y \in \overline{\mathbb{R}^n}$. The **spherical (chordal) diameter** of a set $E \subset \overline{\mathbb{R}^n}$ is

$$h(E) = \sup_{x,y \in E} h(x,y)\,. \tag{10.3}$$

We use further the following statement of the Arzela-Ascoli type, see, e.g., Corollary 7.5. in [115].

Proposition 10.1. *If (X,d) is a separable metric space and (X',d') is a compact metric space, then a family $\mathfrak{F}$ of mappings $f : X \to X'$ is normal if and only if $\mathfrak{F}$ is equicontinuous.*

Combining Theorem 9.1 and Corollaries 9.1–9.3 with the results of the work [152], see also Chapter 7 in [115], we have the following statements.

Theorem 10.1. *Let D and D' be domains in $\mathbb{R}^n$, $n \geqslant 3$, and let $\varphi : [0,\infty) \to [0,\infty)$ be an increasing function such that $\varphi(0) = 0$*

and

$$\int\limits_{1}^{\infty}\left[\frac{t}{\varphi(t)}\right]^{\frac{1}{n-2}}\,dt<\infty\,. \tag{10.4}$$

Let $f:D\to D'$ be a homeomorphism of finite distortion in the Orlicz-Sobolev class $W^{1,\varphi}_{\rm loc}$ such that $h(\overline{\mathbb{R}^n}\setminus f(D))\geqslant\Delta>0$. Then, for every $x_0\in D$ and $x\in B(x_0,\varepsilon(x_0))$, $\varepsilon(x_0)<d(x_0)={\rm dist}(x_0,\partial D)$,

$$h(f(x),f(x_0))\leqslant\frac{\alpha_n}{\Delta}\exp\left\{-\int\limits_{|x-x_0|}^{\varepsilon(x_0)}\frac{dr}{rk_{x_0}^{\frac{1}{n-1}}(r)}\right\} \tag{10.5}$$

where α_n is some constant depending only on n and $k_{x_0}(r)$ is the average of $[K_f(x)]^{n-1}$ over the sphere $|x-x_0|=r$.

Remark 10.1. The estimate (10.5) can be written in the form

$$h(f(x),f(x_0))\leqslant\frac{\alpha_n}{\Delta}\exp\left\{-\,\omega_{n-1}^{\frac{1}{n-1}}\int\limits_{|x-x_0|}^{\varepsilon(x_0)}\frac{dr}{||K_f||_{n-1}(x_0,r)}\right\} \tag{10.6}$$

where $||K_f||_{n-1}(x_0,r)$ is the norm of K_f in the space L^{n-1} over the sphere $|x-x_0|=r$ and ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$.

Corollary 10.2. *The estimates (10.5) and (10.6) hold for homeomorphisms f of finite distortion in the Sobolev classes $W^{1,p}_{\rm loc}$, $p>n-1$. In particular, these estimates hold for homeomorphisms f of finite distortion with $K_f\in L^q_{\rm loc}$ for $q>n-1$.*

Corollary 10.3. *If*

$$k_{x_0}(r)\leqslant\left[\log\frac{1}{r}\right]^{n-1} \tag{10.7}$$

for $r<\varepsilon(x_0)<\min\{1,d(x_0)\}$, then

$$h(f(x),f(x_0))\leqslant\frac{\alpha_n}{\Delta}\,\frac{\log\frac{1}{\varepsilon(x_0)}}{\log\frac{1}{|x-x_0|}} \tag{10.8}$$

for all $x\in B(x_0,\varepsilon(x_0))$.

Corollary 10.4. *If*

$$K_f(x) \leqslant \log \frac{1}{|x-x_0|}, \qquad x \in B(x_0, \varepsilon(x_0)), \tag{10.9}$$

then (10.8) holds in the ball $B(x_0, \varepsilon(x_0))$.

Corollary 10.5. *Let $n \geqslant 3$, $\varphi : [0,\infty) \to [0,\infty)$ be an increasing function, $\varphi(0) = 0$, satisfying (10.4). Let $f : \mathbb{B}^n \to \mathbb{B}^n$, $f(0) = 0$, be a homeomorphism of finite distortion in the class $W^{1,\varphi}_{\mathrm{loc}}$ such that*

$$\int\limits_{\varepsilon<|x|<1} [K_f(x)]^{n-1} \, \frac{dm(x)}{|x|^n} \leqslant c \log \frac{1}{\varepsilon}, \qquad \varepsilon \in (0,1). \tag{10.10}$$

Then

$$|f(x)| \leqslant \gamma_n \cdot |x|^{\beta_n} \tag{10.11}$$

where the constants γ_n and β_n depend only on n.

Theorem 10.2. *Let D and D' be domains in $\mathbb{R}^n$, $n \geqslant 3$, and let $\varphi : [0,\infty) \to [0,\infty)$ be an increasing function, $\varphi(0) = 0$, such that (10.4) holds. Suppose $f : D \to D'$ is a homeomorphism of finite distortion in the class $W^{1,\varphi}_{\mathrm{loc}}$ such that $h(\overline{\mathbb{R}^n} \setminus f(D)) \geqslant \Delta > 0$ and $K_f(x) \leqslant Q(x)$ where $Q^{n-1} \in \mathrm{FMO}(x_0)$. Then*

$$h(f(x), f(x_0)) \leqslant \frac{\alpha_n}{\Delta} \left\{ \frac{\log \frac{1}{\varepsilon_0}}{\log \frac{1}{|x-x_0|}} \right\}^{\beta} \quad \forall\, x \in B(x_0, \varepsilon_0) \tag{10.12}$$

where $\varepsilon_0 < \mathrm{dist}(x_0, \partial D)$ and α_n depends only on n and β depends on the function Q.

Corollary 10.6. *In particular, the estimate (10.12) holds if*

$$\limsup_{\varepsilon \to 0} \fint_{B(x_0,\varepsilon)} Q^{n-1}(x)\, dm(x) < \infty\,. \tag{10.13}$$

Next, let D be a domain in $\mathbb{R}^n$, $n \geqslant 3$, and let $\varphi : [0,\infty) \to [0,\infty)$ be an increasing function, $\varphi(0) = 0$, $Q : D \to [0,\infty]$ be a measurable function. Let $\mathcal{O}^{\varphi}_{Q,\Delta}$ be the class of all homeomorphisms of finite distortion in the Orlicz-Sobolev class $W^{1,\varphi}_{\mathrm{loc}}$ such that $h\left(\overline{\mathbb{R}^n} \setminus f(D)\right)$

$\geqslant \Delta > 0$ and $K_f(x) \leqslant Q(x)$ a.e. Moreover, let $\mathcal{S}^p_{Q,\Delta}$, $p \geqslant 1$, denote the classes $\mathcal{O}^{\varphi}_{Q,\Delta}$ with $\varphi(t) = t^p$. Finally, let $\mathcal{K}^p_{Q,\Delta}$ be the class of all homeomorphisms with finite distortion such that $K_f \in L^p_{\rm loc}$, $p \geqslant 1$, $K_f(x) \leqslant Q(x)$ a.e. and $h\left(\overline{\mathbb{R}^n} \setminus f(D)\right) \geqslant \Delta > 0$.

By Proposition 10.1 the above estimates of distortion now yield:

Theorem 10.3. *Let $\varphi : [0,\infty) \to [0,\infty)$ be an increasing function such that $\varphi(0) = 0$ and (10.4) hold. If $Q^{n-1} \in {\rm FMO}$, then $\mathcal{O}^{\varphi}_{Q,\Delta}$ is a normal family.*

Corollary 10.7. *Under (10.4) the class $\mathcal{O}^{\varphi}_{Q,\Delta}$ is normal if*

$$\overline{\lim_{\varepsilon\to 0}} \; -\!\!\!\!\!\!\int_{B(x_0,\varepsilon)} Q^{n-1}(x)\, dm(x) < \infty \quad \forall\, x_0 \in D. \tag{10.14}$$

Corollary 10.8. *In particular, the classes $\mathcal{S}^p_{Q,\Delta}$ and $\mathcal{K}^p_{Q,\Delta}$ are normal under $p > n-1$ if either $Q^{n-1} \in {\rm FMO}$ or (10.14) holds.*

Theorem 10.4. *Let $\Delta > 0$ and $Q : D \to [0,\infty]$ be a measurable function such that*

$$\int\limits_0^{\varepsilon(x_0)} \frac{dr}{||Q||_{n-1}(x_0,r)} = \infty \quad \forall\, x_0 \in D \tag{10.15}$$

where $\varepsilon(x_0) < {\rm dist}(x_0, \partial D)$ and $||Q||_{n-1}(x_0,r)$ denotes the norm of Q in L^{n-1} over the sphere $|x - x_0| = r$. Then the classes $\mathcal{O}^{\varphi}_{Q,\Delta}$, $\mathcal{S}^p_{Q,\Delta}$, $\mathcal{K}^p_{Q,\Delta}$ form normal families if φ satisfies (10.4), correspondingly, $p > n-1$.

Corollary 10.9. *The classes $\mathcal{O}^{\varphi}_{Q,\Delta}$, $\mathcal{S}^p_{Q,\Delta}$, $\mathcal{K}^p_{Q,\Delta}$ form normal families if φ satisfies (10.4), correspondingly, $p > n-1$ and $Q(x)$ has singularities only of the logarithmic type.*

Let D be a fixed domain in the extended space $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$, $n \geqslant 3$, $\varphi : [0,\infty) \to [0,\infty)$ be an increasing function, $\varphi(0) = 0$. Given a function $\Phi : [0,\infty] \to [0,\infty]$, $M > 0$, $\Delta > 0$, $\mathcal{O}^{\Phi,\varphi}_{M,\Delta}$ denotes

the collection of all homeomorphisms of finite distortion in the Orlicz-Sobolev class $W^{1,\varphi}_{\rm loc}$ such that $h\left(\overline{\mathbb{R}^n}\setminus f(D)\right)\geqslant\Delta>0$ and

$$\int\limits_{D}\Phi\left(K_f^{n-1}(x)\right)\frac{dm(x)}{(1+|x|^2)^n}\ \leqslant\ M\,. \tag{10.16}$$

Similarly, $\mathcal{S}^{\Phi,p}_{M,\Delta}$, $p\geqslant 1$, denote the classes $\mathcal{O}^{\Phi,\varphi}_{M,\Delta}$ with $\varphi(t)=t^p$. Finally, let $\mathcal{K}^{\Phi,p}_{M,\Delta}$, $p\geqslant 1$, be the class of all homeomorphisms with finite distortion such that $K_f\in L^p_{\rm loc}$, $p\geqslant 1$, (10.16) holds for K_f and $h\left(\overline{\mathbb{R}^n}\setminus f(D)\right)\geqslant\Delta>0$.

Combining Theorem 9.1, Corollaries 9.1–9.3 and also Theorem 6.1 under $p=n-1$, we have the following statements, cf. [153].

Theorem 10.5. *Let $\Phi:[0,\infty]\to[0,\infty]$ be a convex increasing function such that*

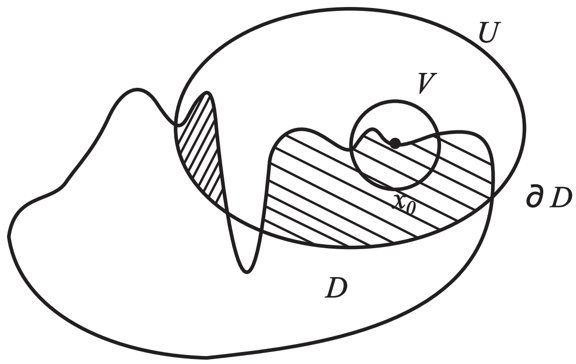
$$\int\limits_{\delta_0}^{\infty}\frac{d\tau}{\tau\left[\Phi^{-1}(\tau)\right]^{\frac{1}{n-1}}}=\infty \tag{10.17}$$

for some $\delta_0>\Phi(0)$. Then the classes $\mathcal{O}^{\Phi,\varphi}_{M,\Delta}$ under (10.4) and $\mathcal{S}^{\Phi,p}_{M,\Delta}$ and $\mathcal{K}^{\Phi,p}_{M,\Delta}$ under $p>n-1$ are equicontinuous and, consequently, form normal families of mappings for every $M\in(0,\infty)$ and $\Delta\in(0,1)$.

Remark 10.2. As it follows from [153], the condition (10.17) is not only sufficient but also necessary for normality of the given classes. Moreover, by Proposition 6.1 we may use instead of (10.17) each of the equivalent conditions (6.4)–(6.9) under $p=n-1$.

11 On domains with regular boundaries

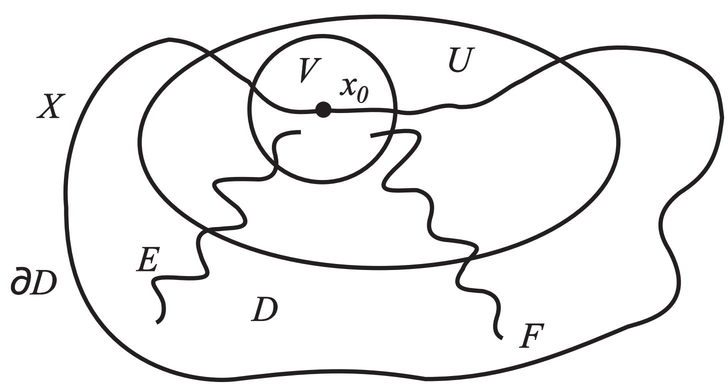
Recall first of all the following topological notion. A domain $D\subset\mathbb{R}^n$, $n\geqslant 2$, is said to be **locally connected at a point** $x_0\in\partial D$ if, for every neighborhood U of the point x_0, there is a neighborhood $V\subseteq U$ of x_0 such that $V\cap D$ is connected. Note that every Jordan domain D in $\mathbb{R}^n$ is locally connected at each point of ∂D, see, e.g., [179], p. 66.



We say that ∂D is **weakly flat at a point** $x_0 \in \partial D$ if, for every neighborhood U of the point x_0 and every number $P > 0$, there is a neighborhood $V \subset U$ of x_0 such that

$$M(\Delta(E, F; D)) \geqslant P \tag{11.1}$$

for all continua E and F in D intersecting ∂U and ∂V. Here and later on, $\Delta(E, F; D)$ denotes the family of all paths $\gamma : [a, b] \to \overline{\mathbb{R}^n}$ connecting E and F in D, i.e., $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in D$ for all $t \in (a, b)$. We say that the boundary ∂D is **weakly flat** if it is weakly flat at every point in ∂D.



We also say that a point $x_0 \in \partial D$ is **strongly accessible** if, for every neighborhood U of the point x_0, there exist a compactum E in D, a neighborhood $V \subset U$ of x_0 and a number $\delta > 0$ such that

$$M(\Delta(E, F; D)) \geqslant \delta \tag{11.2}$$

for all continua F in D intersecting ∂U and ∂V. We say that the boundary ∂D is **strongly accessible** if every point $x_0 \in \partial D$ is strongly accessible.

Here, in the definitions of strongly accessible and weakly flat boundaries, one can take as neighborhoods U and V of a point x_0 only balls (closed or open) centered at x_0 or only neighborhoods of x_0 in another fundamental system of neighborhoods of x_0. These conceptions can also be extended in a natural way to the case of $\overline{\mathbb{R}^n}$ and $x_0 = \infty$. Then we must use the corresponding neighborhoods of ∞.

It is easy to see that if a domain D in $\mathbb{R}^n$ is weakly flat at a point $x_0 \in \partial D$, then the point x_0 is strongly accessible from D. Moreover, it was proved by us that if a domain D in $\mathbb{R}^n$ is weakly flat at a point $x_0 \in \partial D$, then D is locally connected at x_0, see, e.g., Lemma 5.1 in [81] or Lemma 3.15 in [115].

The notions of strong accessibility and weak flatness at boundary points of a domain in $\mathbb{R}^n$ defined in [80] are localizations and generalizations of the corresponding notions introduced in [113]–[114], cf. with the properties P_1 and P_2 by Väisälä in [173] and also with the quasiconformal accessibility and the quasiconformal flatness by Näkki in [121]. Many theorems on a homeomorphic extension to the boundary of quasiconformal mappings and their generalizations are valid under the condition of weak flatness of boundaries. The condition of strong accessibility plays a similar role for a continuous extension of the mappings to the boundary. In particular, recently we have proved the following significant statements, see either Theorem 10.1 (Lemma 6.1) in [81] or Theorem 9.8 (Lemma 9.4) in [115].

Proposition 11.1. *Let D and D' be bounded domains in $\mathbb{R}^n$, $n \geqslant 2$, $Q : D \to (0,\infty)$ a measurable function and $f : D \to D'$ a lower Q-homeomorphism on ∂D. Suppose that the domain D is locally connected on ∂D and that the domain D' has a (strongly accessible) weakly flat boundary. If*

$$\int\limits_0^{\delta(x_0)} \frac{dr}{||\,Q||_{n-1}(x_0,r)} = \infty \qquad \forall\ x_0 \in \partial D \tag{11.3}$$

for some $\delta(x_0) \in (0, d(x_0))$ *where* $d(x_0) = \sup\limits_{x \in D} |x - x_0|$ *and*

$$\|\, Q\|_{n-1}(x_0, r) = \left(\int\limits_{D \cap S(x_0,r)} Q^{n-1}(x)\, d\mathcal{A} \right)^{\frac{1}{n-1}} ,$$

then f *has a (continuous) homeomorphic extension* $\overline{f}$ *to* $\overline{D}$ *that maps* $\overline{D}$ *(into) onto* $\overline{D'}$*.*

Here as usual $S(x_0, r)$ denotes the sphere $|x - x_0| = r$ and the closure is understood in the sense of the extended space $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$.

A domain $D \subset \mathbb{R}^n$ is called a **quasiextremal distance domain**, abbr. **QED-domain**, see [37], if

$$M(\Delta(E, F; \overline{\mathbb{R}^n}) \leqslant K \cdot M(\Delta(E, F; D)) \tag{11.4}$$

for some $K \geqslant 1$ and all pairs of nonintersecting continua E and F in D.

It is well known, see e.g. Theorem 10.12 in [173], that

$$M(\Delta(E, F; \mathbb{R}^n)) \geqslant c_n \log \frac{R}{r} \tag{11.5}$$

for any sets E and F in $\mathbb{R}^n$, $n \geqslant 2$, intersecting all the circles $S(x_0, \rho)$, $\rho \in (r, R)$. Hence a QED-domain has a weakly flat boundary. One example in [115], Section 3.8, shows that the inverse conclusion is not true even among simply connected plane domains.

A domain $D \subset \mathbb{R}^n$, $n \geqslant 2$, is called a **uniform domain** if each pair of points x_1 and $x_2 \in D$ can be joined with a rectifiable curve γ in D such that

$$s(\gamma) \;\leqslant\; a \cdot |\, x_1 - x_2| \tag{11.6}$$

and

$$\min_{i=1,2} \; s(\gamma(x_i, x)) \;\leqslant\; b \cdot d(x, \partial D) \tag{11.7}$$

for all $x \in \gamma$ where $\gamma(x_i, x)$ is the portion of γ bounded by x_i and x, see [117]. It is known that every uniform domain is a QED-domain

but there exist QED-domains that are not uniform, see [37]. Bounded convex domains and bounded domains with smooth boundaries are simple examples of uniform domains and, consequently, QED-domains as well as domains with weakly flat boundaries.

A closed set $X \subset \mathbb{R}^n$, $n \geqslant 2$, is called a **null-set for extremal distances**, abbr. **NED-set**, if

$$M(\Delta(E, F; \mathbb{R}^n)) = M(\Delta(E, F; \mathbb{R}^n \setminus X)) \tag{11.8}$$

for any two nonintersecting continua E and $F \subset \mathbb{R}^n \setminus X$.

Remark 11.1. It is known that if $X \subset \mathbb{R}^n$, $n \geqslant 2$, is a NED-set, then

$$|X| = 0 \tag{11.9}$$

and X does not locally disconnect $\mathbb{R}^n$, i.e., see [61],

$$\dim X \leqslant n - 2\,, \tag{11.10}$$

and, conversely, if a set $X \subset \mathbb{R}^n$ is closed and

$$H^{n-1}(X) = 0\,, \tag{11.11}$$

then X is a NED-set, see [175]. Note also that the complement of a NED-set in $\mathbb{R}^n$ is a very particular case of a QED-domain.

Further we denote by $C(X, f)$ the **cluster set** of the mapping $f : D \to \overline{\mathbb{R}^n}$ for a set $X \subset \overline{D}$,

$$C(X, f) \ : \ = \left\{ y \in \overline{\mathbb{R}^n} : \ y = \lim_{k \to \infty} f(x_k), \ x_k \to x_0 \in X, \ x_k \in D \right\} \ . \tag{11.12}$$

Note that the inclusion $C(\partial D, f) \subseteq \partial D'$ holds for every homeomorphism $f : D \to D'$, see, e.g., Proposition 13.5 in [115].

12 The boundary behavior

In this section we assume that $\varphi : [0, \infty) \to [0, \infty)$ is an increasing function with $\varphi(0) = 0$ such that

$$\int\limits_1^{\infty} \left[\frac{t}{\varphi(t)} \right]^{\frac{1}{n-2}} dt < \infty \ . \tag{12.1}$$

In view of Theorem 9.1, we have by Proposition 11.1 the following statement.

Theorem 12.1. *Let D and D' be bounded domains in $\mathbb{R}^n$, $n \geqslant 3$, and let $f : D \to D'$ be a homeomorphism of finite distortion in $W^{1,\varphi}_{\rm loc}$ with the condition (12.1). Suppose that the domain D is locally connected on ∂D and that the domain D' has a (strongly accessible) weakly flat boundary. If*

$$\int\limits_{0}^{\delta(x_0)} \frac{dr}{||\, K_f||_{n-1}(x_0, r)} \;=\; \infty \qquad \forall\; x_0 \in \partial D \tag{12.2}$$

for some $\delta(x_0) \in (0, d(x_0))$ where $d(x_0) = \sup\limits_{x\in D} |\, x - x_0|$ and

$$||\, K_f||_{n-1}(x_0, r) = \left(\int\limits_{D\cap S(x_0,r)} K_f^{n-1}(x)\, d\mathcal{A} \right)^{\frac{1}{n-1}},$$

then f has a (continuous) homeomorphic extension $\overline{f}$ to $\overline{D}$ that maps $\overline{D}$ (into) onto $\overline{D'}$.

In particular, as a consequence of Theorem 12.1 we obtain the following generalization of the well-known Gehring-Martio theorem on a homeomorphic extension to the boundary of quasiconformal mappings between QED domains, cf. [37].

Corollary 12.1. *Let D and D' be bounded domains with weakly flat boundaries in $\mathbb{R}^n$, $n \geqslant 3$, and let $f : D \to D'$ be a homeomorphism of finite distortion in D in the class $W^{1,p}_{\rm loc}$, $p > n-1$, in particular, $K_f \in L^q_{\rm loc}$, $q > n-1$. If the condition (12.2) holds at every point $x_0 \in \partial D$, then f has a homeomorphic extension to $\overline{D}$.*

The continuous extenxion to the boundary of the inverse mappings has a simpler criterion. Namely, in view of Theorem 9.1, we have by Theorem 9.1 in [81] or Theorem 9.6 in [115] the next statement.

Theorem 12.2. *Let D and D' be domains in $\mathbb{R}^n$, $n \geqslant 3$, D be locally connected on ∂D and $\partial D'$ be weakly flat. If f is a homeomorphism of finite distortion of D onto D' in the class $W^{1,\varphi}_{\rm loc}$ with the condition (12.1) and $K_f \in L^{n-1}(D)$, then f^{-1} has an extension to $\overline{D'}$ by continuity in $\overline{\mathbb{R}^n}$.*

However, as it follows from the example in Proposition 6.3 from [115], any degree of integrability $K_f \in L^q(D)$, $q \in [1, \infty)$, cannot guarantee the extension by continuity to the boundary of the direct mappings.

Similarly, in view of Theorem 9.1, we have by Theorem 8.1 in [81] or Theorem 9.5 in [115] the next result.

Theorem 12.3. *Let D be a domain in $\mathbb{R}^n$, $n \geq 3$, $X \subset D$, and let f be a homeomorphism with finite distortion of $D \backslash X$ into $\overline{\mathbb{R}^n}$ in $W^{1,\varphi}_{\rm loc}$ with the condition (12.1). Suppose that X and $C(X, f)$ are* NED *sets. If*

$$\int\limits_0^{\varepsilon(x_0)} \frac{dr}{||\, K_f||_{n-1}(x_0, r)} \;=\; \infty \qquad\qquad \forall\ x_0 \in \partial D \tag{12.3}$$

where $0 < \varepsilon_0 < d_0 = \mathrm{dist}\,(x_0, \partial D)$ *and*

$$||\, K_f||_{n-1}(x_0, r) \;=\; \left(\int\limits_{|x-x_0|=r} K_f^{n-1}(x)\, d\mathcal{A} \right)^{\frac{1}{n-1}}, \tag{12.4}$$

then f is extended by continuity in $\overline{\mathbb{R}^n}$ to D.

Remark 12.1. In particular, the conclusion of Theorem 12.3 is valid if X is a closed set with

$$H^{n-1}(X) \;=\; 0 \;=\; H^{n-1}(C(X, f)). \tag{12.5}$$

Finally, in view of Theorem 9.1, by Theorem 12.1 as well as by Theorem 6.1 under $p = n - 1$ we obtain the following result.

Theorem 12.4. *Let D and D' be bounded domains in $\mathbb{R}^n$, $n \geqslant 3$, D be locally connected on ∂D and D' have (strongly accessible) weakly flat boundary. Suppose $f : D \to D'$ is a homeomorphism of finite distortion in D in the class $W^{1,\varphi}_{\rm loc}$ with the condition (12.1) such that*

$$\int\limits_{D} \Phi(K_f^{n-1}(x))\, dm(x) < \infty \tag{12.6}$$

for a convex increasing function $\Phi : [0, \infty] \to [0, \infty]$. If

$$\int\limits_{\delta_0}^{\infty} \frac{d\tau}{\tau\left[\Phi^{-1}(\tau)\right]^{\frac{1}{n-1}}} = \infty \tag{12.7}$$

for some $\delta_0 > \Phi(0)$, then f has a (continuous) homeomorphic extension $\overline{f}$ to $\overline{D}$ that maps $\overline{D}$ (into) onto $\overline{D'}$.

Remark 12.2. Note that by Theorem 5.1 and Remark 5.1 in [83] the conditions (12.7) are not only sufficient but also necessary for continuous extension to the boundary of f with the integral constraints (12.6).

Recall that by Proposition 6.1 the condition (12.7) is equivalent to each of the conditions (6.4)–(6.9) under $p = n - 1$ and, in particular, to the following condition

$$\int\limits_{\delta}^{\infty} \log\, \Phi(t)\, \frac{dt}{t^{n'}} \;=\; +\infty \tag{12.8}$$

for some $\delta > 0$ where $\frac{1}{n'} + \frac{1}{n} = 1$, i.e., $n' = 2$ for $n = 2$, n' is strictly decreasing in n and $n' = n/(n-1) \to 1$ as $n \to \infty$.

Finally note that all the results in this section hold, in particular, if $f \in W^{1,p}_{\rm loc}$, $p > n - 1$ and, in particular, if $K_f \in L^q_{\rm loc}$, $q > n - 1$, and, in particular, if D and D' are either bounded convex domains or bounded domains with smooth boundaries.

13 Some examples

The following lemma is a base for demonstrating preciseness of the Calderon type conditions in the above results.

Lemma 13.1. *Let $\varphi:[0,\infty)\to[0,\infty)$ be a convex increasing function such that*

$$\int\limits_{1}^{\infty}\left[\frac{t}{\varphi(t)}\right]^{\frac{1}{k-1}}dt=\infty \tag{13.1}$$

for a natural number $k\geqslant 2$. Then there is an embedding g of $\mathbb{R}^k$ into $\mathbb{R}^{k+1}$ of the form $g(x)=(x,f(x))$, such that $g\in W^{1,\varphi}_{\rm loc}$ but g has not (N)-property with respect to k-dimensional Hausdorff measure.

Corollary 13.1. *For every $k\geqslant 2$, there is an embedding g of $\mathbb{R}^k$ into $\mathbb{R}^{k+1}$ of the form $g(x)=(x,f(x))$ in the class $W^{1,k}_{\rm loc}$ that has not (N)-property with respect to k-dimensional Hausdorff measure.*

Remark 13.1. The corresponding examples of embeddings g in the class $W^{1,k}_{\rm loc}$ for $k=2$ from $\mathbb{R}^2$ into $\mathbb{R}^3$ based on the theory of conformal mappings are known long ago, see, e.g., [142] and [145]. However, they have not the form $g(x)=(x,f(x))$ and cannot be applied for constructing examples of homeomorphisms in the class $W^{1,2}_{\rm loc}$ from $\mathbb{R}^3$ into $\mathbb{R}^3$ as in Theorem 13.1 and Corollary 13.2 further.

Proof of Lemma 13.1. We apply further for our purposes a little modified construction of Calderon in [16], p. 210-211. Let F and F_* be functions from Proposition 2.4 corresponding to the function $\varphi_*(t)=\varphi(t+k)-\varphi(k)$. It is clear that φ_* satisfies (13.1), too.

Let us give 3 decreasing sequences of positive numbers r_l, ϱ_l and ϱ_l^*, $l=1,2,\ldots$, by induction in the following way. Set r_1 is equal to the maximal number $r>0$ such that $r\leqslant 2^{-2}$ and

$$\int\limits_{|x|\leqslant r}\varphi_*\left(|\nabla F_*|\right)dm(x)\leqslant 2^{-k}.$$

The numbers ϱ_1 and ϱ_1^* are defined from the equalities $F(\varrho_1) = F(r_1)+1$ and $F(\varrho_1^*) = F(r_1)+3/4$, correspondingly. If the numbers $r_1, \ldots, r_{l-1}$, $\varrho_1, \ldots, \varrho_{l-1}$ and $\varrho_1^*, \ldots, \varrho_{l-1}^*$ have been given, then we set r_l is equal to the maximal number $r > 0$ such that

$$\int\limits_{|x|\leqslant r} \varphi_* \left(|\nabla F_*|\right) dm(x) \leqslant 2^{-lk} \tag{13.2}$$

and, moreover,

$$r \leqslant \min\{\, \varrho_{l-1}, 2^{-2l}(\varrho_{l-1}^* - \varrho_{l-1}), 1/[2^{l+2}(l-1)\,|F'(\varrho_{l-1})|]\,\}\,. \tag{13.3}$$

Then we define ϱ_l and ϱ_l^* from the equalities $F(\varrho_l) = F(r_l)+1$ and $F(\varrho_l^*) = F(r_l)+3/4$, respectively. Note that by monotonicity of the derivative $|F'(\varrho_{l-1})| = \max\{|F'(t)| : t \geqslant \varrho_{l-1}\}$. It is also clear by the construction that $\varrho_l < \varrho_l^* < r_l$ and that the sequence $\varrho_l^* - \varrho_l < r_l$ is decreasing because the function $F'(t)$ is non-decreasing.

Setting $F_l(r) = \min\,[\,1, F(r) - F(r_l)\,]$ for $r \in [0, r_l]$ and $F_l(r) \equiv 0$ for $r > r_l$, we see that F_l satisfies a uniform Lipschitz condition, that $F_l(0) = 1$ and by (13.2)

$$\int\limits_{\mathbb{R}^k} \varphi_* \left(|\nabla F_l^*|\right) dm(x) \leqslant 2^{-lk} \tag{13.4}$$

where $F_l^*(x) = F_l(|\,x|)$, $x \in \mathbb{R}^k$, $l = 1, 2, \ldots$.

Now, denote by $x^l_{j_1,\ldots,j_k}$, $l = 1, 2, \ldots$, $j_1, \ldots, j_k = 0, \pm 1, \pm 2, \ldots$, the points in $\mathbb{R}^k$ whose coordinates are integral multiples of 2^{-l} with the natural order in $j_1, \ldots, j_k$ along the corresponding coordinate axes. Let $B^l_{j_1,\ldots,j_k}$ be the closed balls centered at $x^l_{j_1,\ldots,j_k}$ with the radii r_l. Note that by the second condition in (13.3) $r_l \leqslant 2^{-2l}$ and the given balls are disjoint each to other for every fixed $l = 2, \ldots$. Next, define

$$f_l(x) \;=\; \sum_{j_1,\ldots,j_k} F_l\left(|x - x^l_{j_1,\ldots,j_k}|\right),$$

$$f_p^*(x) \;=\; \sum_{l=1}^{p} 2^{-l} f_l(x)$$

and

$$f(x) \;=\; \sum_{l=1}^{\infty} 2^{-l} f_l(x) \;=\; \lim_{p\to\infty} f^*_p(x)\,.$$

By the construction, the nonnegative functions $f_l(x)$, $f^*_p(x)$ and $f(x) \leqslant 1$ are continuous. Moreover, it is easy to estimate their oscillations on the balls $B^p_{j_1,\ldots,j_k}$. In particular,

$$\operatorname*{osc}_{B^p_{j_1,\ldots,j_k}} f^*_{p-1} \leqslant \frac{1}{4}\cdot 2^{-p} \operatorname*{osc}_{B^p_{j_1,\ldots,j_k}} f_p = 2^{-(p+2)} < 2^{-(p-1)}. \tag{13.5}$$

Indeed, by the triangle inequality and the monotonicity of F',

$$\operatorname*{osc}_{B^p_{j_1,\ldots,j_k}} f^*_{p-1} \leqslant \sum_{l=1}^{p-1} 2^{-l} \operatorname*{osc}_{B^p_{j_1,\ldots,j_k}} f_l \leqslant r_p \sum_{l=1}^{p-1} |F'(\varrho_l)| \leqslant r_p(p-1)|F'(\varrho_{p-1})|$$

and, thus, applying the third condition in (13.3), we come to the (13.5).

Let us show that the mapping $g(x) = (x, f(x))$ belongs to the class $W^{1,\varphi}_{\mathrm{loc}}$. To this end, consider an arbitrary closed oriented unit cube C in $\mathbb{R}^k$ whose vertices have irrational coordinates. Note that the cube C contains exactly 2^{lk} points $x^l_{j_1,\ldots,j_k}$. Thus, by periodicity of the picture and the condition (13.4) we have that

$$\int\limits_C \varphi_*\left(|\nabla f_l|\right) dm(x) \leqslant 1 \tag{13.6}$$

and, applying the (discrete) Jensen inequality, see, e.g., Theorem 86 in [49], we obtain that

$$\int\limits_C \varphi_*\left(|\nabla f|\right) dm(x) \leqslant \int\limits_C \varphi_*\left(\frac{\sum\limits_{l=1}^{\infty} 2^{-l}|\nabla f_l|}{\sum\limits_{l=1}^{\infty} 2^{-l}}\right) dm(x) \leqslant$$

$$\leqslant \sum_{l=1}^{\infty} 2^{-l} \int\limits_C \varphi_*\left(|\nabla f_l|\right) dm(x) \leqslant 1\,,$$

Finally, since $|\nabla g| = \sqrt{k + |\nabla f|^2} \leqslant k + |\nabla f|$, we have that

$$\int\limits_C \varphi\left(|\nabla g|\right) dm(x) \leqslant 1 + \varphi(k)\,. \tag{13.7}$$

Next, let us fix a closed oriented unit cube C_0 in $\mathbb{R}^k$ whose center has irrational coordinates and let E_l, $l = 1, 2, \ldots$, be the union of all balls $B^l_{j_1,\ldots,j_k}$ centered at points $x^l_{j_1,\ldots,j_k}$ in the cube C_0. By the second condition in (13.3) we have that $|E_l| \leqslant 2^{lk} \cdot \Omega_k \cdot 2^{-2lk} = \Omega_k 2^{-lk}$ where Ω_k is the volume of the unit ball in $\mathbb{R}^k$. Setting $\mathcal{E}_m = \bigcup\limits_{l=m}^{\infty} E_l$, $m = 1, 2 \ldots$, we see that $|\mathcal{E}_m| \leqslant \frac{\Omega_k}{2^k - 1}\, 2^{-k(m-1)} \to 0$ as $m \to \infty$, i.e., the set $\mathcal{E} = \bigcap\limits_{m=1}^{\infty} \mathcal{E}_m$ is of the Lebesgue measure zero in $\mathbb{R}^k$. Similarly, $\mu_{k-1}\left(\mathrm{pr}_i\, E_l\right) \leqslant \Omega_{k-1} 2^{-l(k-1)}$ and $\mu_{k-1}\left(\mathrm{pr}_i\, \mathcal{E}_m\right) \leqslant \frac{\Omega_{k-1}}{2^{k-1}-1}\, 2^{-(k-1)(m-1)} \to 0$ as $m \to \infty$, i.e., $\mu_{k-1}(\mathrm{pr}_i\, \mathcal{E}) = 0$ where pr_i denotes the projection into the coordinate hyperplane P_i which is perpendicular to the i-th coordinate axis, $i = 1, 2, \ldots, k$ in $\mathbb{R}^k$ and μ_{k-1} is the $(k-1)$-dimensional Lebesgue measure on P_i.

Let us prove that every straight line segment L in the cube C which is parallel to a coordinate axis, say to the axis Ox_1, and does not intersect the set $\mathcal{E}$ has only a finite number of joint points with the spheres $\partial B^l_{j_1,\ldots,j_k}$. Indeed, assume that such a segment L intersects an infinite number of the closed balls $B^l_{j_1,\ldots,j_k}$. Recall that the cube C intersects only a finite number of such balls under each fixed $l = 1, 2, \ldots$. Hence there exists an infinite sequence of balls B_{l_m} among $B^{l_m}_{j_1,\ldots,j_k}$ such that $L \cap B_{l_m} \neq \varnothing$, $m = 1, 2, \ldots$ and $l_{m_1} \neq l_{m_2}$ for $m_1 \neq m_2$, i.e., $l_m \to \infty$ as $m \to \infty$. Note that the end points of the segment L can belong only to a finite number of the balls B_{l_m} because in the contrary case it would be $L \cap \mathcal{E} \neq \varnothing$. Thus, we may assume that length $L \cap B_{l_m} > 0$ for all $m = 1, 2, \ldots$. Remark that the distance between the centers $x^{l_m}_{j_1,\ldots,j_k}$ of the balls $B^{l_m}_{j_1,\ldots,j_k}$, $j_1 = 0, \pm 1, \pm 2, \ldots$, as well as between their projections on the straight line of L, is equal to $2^{-l_m} \to 0$ as $m \to \infty$. Hence we may assume without loss of generality that the sequence of the segments $L \cap B_{l_m}$

is monotone decreasing. However, then by the Cantor theorem, see e.g. 4.41.I (2′) in [100], we obtain $\bigcap\limits_{m=1}^{\infty} B_{l_m} \cap L \neq \varnothing$ that contradicts the condition $\mathcal{E} \cap L = \varnothing$.

Thus, g is piecewise monotone and smooth and hence it is absolutely continuous on a.e. segment L in the cube C which is parallel to a coordinate axis. Consequently, g is ACL and by (13.7) $g \in W^{1,\varphi}_{\rm loc}$.

Let us show that the set $E = g(\mathcal{E})$ in $\mathbb{R}^{k+1}$ is not of k-dimensional Hausdorff measure zero.

Given a closed oriented unit cube C_* in $\mathbb{R}^k$ whose center has irrational coordinates and whose edge length $L = 2^{-m}$ for some positive integer m, $l \geqslant m$, we have that

$$\sum \left[d\left(f(B^l_{j_1,\ldots,j_k})\right)\right]^k \leqslant 2^{3k}L^k \tag{13.8}$$

where the sum is taken over all balls $B^l_{j_1,\ldots,j_k}$ whose centers $x^l_{j_1,\ldots,j_k}$ belong to the cube C_*. Indeed, C_* contains exactly $2^{(l-m)k}$ points $x^l_{j_1,\ldots,j_k}$. Moreover, every set $f(B^l_{j_1,\ldots,j_k})$ is contained in a cylinder whose base radius is less or equal to 2^{-2l}, see (13.3), and whose height is less or equal to $2^{-(l-1)} + 2^{-l} + \ldots = 2^{-(l-2)}$, see (13.5). Hence

$$d\left(f(B^l_{j_1,\ldots,j_k})\right) \leqslant \sqrt{2^{-2(l-2)} + 2^{2(-2l+1)}} =$$
$$= 2^{-(l-2)} \cdot \sqrt{1 + 2^{-2(l+1)}} \leqslant 2^{-(l-3)} = 8 \cdot 2^{-l}$$

that implies (13.8).

Now, let us prove the following lower estimate of the diameters of the images of the balls $B^p_{j_1,\ldots,j_k}$:

$$d\left(f(B^p_{j_1,\ldots,j_k})\right) \geqslant 2^{-(p+1)}. \tag{13.9}$$

It is sufficient for this purpose to show that

$$\operatorname*{osc}_{B^p_{j_1,\ldots,j_k}} f \geqslant 2^{-(p+1)}$$

and, in turn, it suffices to demonstrate that

$$\operatorname*{osc}_{L^p_{j_1,\dots,j_k}} f \geqslant 2^{-(p+1)}$$

where $L^p_{j_1,\dots,j_k}$ is the intersection of the ball $B^p_{j_1,\dots,j_k}$ with the straight line L passing through its center $x^p_{j_1,\dots,j_k}$ parallely to one of the coordinate axes. Indeed, by the condition (13.3), the length of the intersection of the line L with the set $\mathcal{E}_{p+1}$ can be easy estimated:

$$\text{length}\,(L \cap \mathcal{E}_{p+1}) \leqslant \sum_{l=p+1}^{\infty} 2r_l \cdot 2^l \leqslant 2 \sum_{l=p+1}^{\infty} 2^{-2l}(\varrho^*_{l-1} - \varrho_{l-1}) \cdot 2^l \leqslant$$

$$\leqslant 2(\varrho^*_p - \varrho_p) \sum_{l=p+1}^{\infty} 2^{-l} \leqslant 2^{-(p-1)}(\varrho^*_p - \varrho_p) \leqslant \varrho^*_p - \varrho_p \,.$$

Hence by the choice of the number ϱ^*_p

$$\operatorname*{osc}_{\Delta^p_{j_1,\dots,j_k}} f_p \;\geqslant\; \frac{3}{4} \operatorname*{osc}_{B^p_{j_1,\dots,j_k}} f_p$$

where $\Delta^p_{j_1,\dots,j_k} = L^p_{j_1,\dots,j_k} \setminus \mathcal{E}_{p+1}$. Thus, by the condition (13.5) and the triangle inequality

$$\operatorname*{osc}_{\Delta^p_{j_1,\dots,j_k}} f \;=\; \operatorname*{osc}_{\Delta^p_{j_1,\dots,j_k}} f^*_p \;\geqslant\; \operatorname*{osc}_{\Delta^p_{j_1,\dots,j_k}} f_p - \operatorname*{osc}_{\Delta^p_{j_1,\dots,j_k}} f^*_{p-1} \;\geqslant$$

$$\geqslant\; \frac{1}{2} \cdot 2^{-p} \operatorname*{osc}_{B^p_{j_1,\dots,j_k}} f_p \;=\; 2^{-(p+1)}$$

and the lower estimate (13.9) follows.

Finally, let $\varepsilon > 0$ and let $\{A_j\}$ be a cover of E such that $d(A_j) < \varepsilon$, $j = 1, 2, \dots$. Note that for each A_j there is a closed oriented cube C_j such that $A_j \subseteq C_j$ and whose edge length L_j is less or equal to $d(A_j)$. However, it is more convenient to use closed oriented cubes with $L_j = 2^{-m_j}$ for some positive integer m_j such that $L_j \leqslant 2d(A_j)$. Let $\mathbb{N}$ be the collection of all positive integers. Set for arbitrary $l \in \mathbb{N}$

$$S_l = \left\{(j_1, \dots, j_k) : \; x^l_{j_1,\dots,j_k} \in C_0\right\} \;, \quad J_l = \{j \in \mathbb{N} : m_j \leqslant l\} \;,$$

and

$$S_l^* = \left\{ (j_1, \ldots, j_k) \in S_l : \ x^l_{j_1,\ldots,j_k} \in \bigcup_{j \in J_l} \operatorname{pr} C_j \right\} .$$

Here pr denotes the natural projection from $\mathbb{R}^{k+1}$ into $\mathbb{R}^k$. Thus, we have by (13.8) that for every $l \in \mathbb{N}$

$$2^k \sum_{j=1}^{\infty} [d(A_j)]^k \ \geqslant \ \sum_{j=1}^{\infty} L_j^k \ \geqslant \ 2^{-3k} \sum_{(j_1,\ldots,j_k)\in S_l^*} \left[d\left(f(B^l_{j_1,\ldots,j_k}) \right) \right]^k .$$

Denote by N_l and N_l^* the numbers of indexes $(j_1, \ldots, j_k)$ in S_l and S_l^*, correspondingly. Note that by the construction the ratio N_l^*/N_l is non-decreasing and it converges to 1 as $l \to \infty$ because $\{C_j\}$ covers E, consequently, $\{\operatorname{pr} C_j\}$ covers $\mathcal{E}$ and hence $\bigcup\limits_{j=1}^{\infty} \operatorname{pr} C_j$ includes all points $x^l_{j_1,\ldots,j_k}$ with $(j_1, \ldots, j_k) \in S_l$. However, S_l contains exactly 2^{lk} indexes $(j_1, \ldots, j_k)$ and by (13.9) $d\left(f(B^l_{j_1,\ldots,j_k})\right) \geqslant 2^{-(l+1)}$. Consequently,

$$\sum_{j=1}^{\infty} [d(A_j)]^k \geqslant 2^{-5k}$$

and, thus, $H^k(E) \geqslant 2^{-5k} > 0$ in view of arbitrariness of $\varepsilon > 0$. The proof is complete.

Remark 13.2. It is known that each homeomorphism of $\mathbb{R}^k$ onto itself in the class $W^{1,k}_{\text{loc}}$ has the (N)-property, see Lemma III.6.1 in [104] for $k = 2$ and [147] for $k > 2$. The same is valid also for open mappings, see [107]. On the other hand, there exist examples of homeomorphisms $W^{1,p}_{\text{loc}}$ for all $p < k$ that have not the (N)-property, see [133]. Moreover, Cezari in [17] proved that continuous plane mappings $f : D \to \mathbb{R}^2$ in the class ACL^p, $p > 2$, has the (N)-property and that there exist examples of such mappings in ACL^2 that have not the (N)-property.

Applying the oblique projection $h(x) = (x_1, \ldots, x_{k-1}, x_k + f(x)/4)$ of the surface $g(x) = (x, f(x))$, $x \in \mathbb{R}^k$, onto $\mathbb{R}^k$, $k \geqslant 2$, from

Lemma 13.1, we obtain the corresponding examples of the continuous mappings h_k^φ of $\mathbb{R}^k$ onto itself in the class $W^{1,\varphi}_{\mathrm{loc}}$ that have not the (N)-property for convex increasing functions φ satisfying (13.1). In particular, we obtain in this way the example of a continuous mapping $h_k : \mathbb{R}^k \to \mathbb{R}^k$ in the class $W^{1,k}_{\mathrm{loc}}$ for each integer $k \geqslant 2$ without the (N)-property.

Setting $H(x,y) = h^\varphi_{n-1}(x)$, $x \in \mathbb{R}^{n-1}$, $y \in \mathbb{R}$, $n \geqslant 3$, we obtain examples of continuous mappings $H : \mathbb{R}^n \to \mathbb{R}^{n-1}$ in the class $W^{1,\varphi}_{\mathrm{loc}}$ without the (N)-property with respect to the $(n-1)$-dimensional Hausdorff measure on a.e. hyperplane for each convex increasing function $\varphi : [0,\infty) \to [0,\infty)$ satisfying the condition (13.10) further.

Theorem 13.1. *Let $\varphi : [0,\infty) \to [0,\infty)$ be a convex increasing function such that*

$$\int\limits_1^\infty \left[\frac{t}{\varphi(t)}\right]^{\frac{1}{n-2}} dt = \infty \tag{13.10}$$

for a natural number $n \geqslant 3$. Then there is a homeomorphism H of $\mathbb{R}^n$ onto $\mathbb{R}^n$ of the form $H(x,y) = (x, y + f(x))$, $x \in \mathbb{R}^{n-1}$, $y \in \mathbb{R}$, such that $H \in W^{1,\varphi}_{\mathrm{loc}}$ but H has not (N)-property with respect to $(n-1)$-dimensional Hausdorff measure on any hyperplane $y = \mathrm{const}$.

Proof of Theorem 13.1. Indeed, the function $\varphi_*(t) := \varphi(t+1)$ satisfies (13.10). Set $H(x,y) = g(x) + (0,\dots,0,y) = (x, y + f(x))$, $x \in \mathbb{R}^{n-1}$, $y \in \mathbb{R}$, where $g(x) = (x, f(x))$ is the mapping in Lemma 13.1 under $k = n-1$ corresponding to the function φ_*. Then $|\nabla H| \leqslant 1 + |\nabla g|$ and by monotonicity of φ we have that $H \in W^{1,\varphi}_{\mathrm{loc}}$ because $g \in W^{1,\varphi_*}_{\mathrm{loc}}$.

Corollary 13.2. *For every $n \geqslant 3$, there is a homeomorphism of $\mathbb{R}^n$ onto $\mathbb{R}^n$ of the class $W^{1,n-1}_{\mathrm{loc}}$ of the form $H(x,y) = (x, y + f(x))$, $x \in \mathbb{R}^{n-1}$, $y \in \mathbb{R}$, without (N)-property with respect to $(n-1)$-dimensional Hausdorff measure on any hyperplane $y = \mathrm{const}$.*

Remark 13.3. Note that $\mathbb{R}^n$ can be in the natural way embedded into $\mathbb{R}^m$ for each $m > n$. Thus, by Theorems 4.2 and 13.1

and Remark 13.2, the Calderon type condition (4.5) is not only sufficient but also necessary for continuous mappings $f : \mathbb{R}^n \to \mathbb{R}^m$, $n \geqslant 3$, $m \geqslant n-1$, in the Orlicz-Sobolev classes $W^{1,\varphi}_{\rm loc}$ to have the (N)-property with respect to $(n-1)$-dimensional Hausdorff measure on a.e. hyperplane. Furthermore, Theorem 13.1 shows that the necessity of the condition (4.5) is valid for $m = n$ even for homeomorphisms f. In this connection note also that Corollaries 13.2 disproves Theorem 1.3 from the preprint [23].

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